Adiabatic models of the cosmological radiative era.

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We consider a generalization of the Lemaitre-Tolman-Bondi (LTB) solutions by keeping the LTB metric but replacing its dust matter source by an imperfect fluid with anisotropic pressure Π_{ab}. Assuming that total matter-energy density ρ is the sum of a rest mass term, $\rho^{(m)}$, plus a radiation $\rho^{(r)} = 3p$ density where p is the isotropic pressure, Einstein's equations are fully integrated without having to place any previous assumption on the form of Π_{ab}. Three particular cases of interest are contained: the usual LTB dust solutions (the dust limit), a class of FLRW cosmologies (the homogeneous limit) and of the Vaydia solution (the vacuum limit). Initial conditions are provided in terms of suitable averages and contrast functions of the initial densities of $\rho^{(m)}$, $\rho^{(r)}$ and the 3-dimensional Ricci scalar along an arbitrary initial surface $t = t_i$. We consider the source of the models as an interactive radiation-matter mixture in local thermal equilibrium that must be consistent with causal Extended Irreversible Thermodynamics (hence Π_{ab} is shear viscosity). Assuming near equilibrium conditions associated with small initial density and curvature contrasts, the evolution of the models is qualitatively similar to that of adiabatic perturbations on a matter plus radiation FLRW background. We show that initial conditions exist that lead to thermodynamically consistent models, but only for the full transport equation of Extended Irreversible Thermodynamics. These interactive mixtures provide a reasonable approximation to a dissipative 'tight coupling' characteristic of radiation-matter mixtures in the radiative pre-decoupling era.

I. INTRODUCTION

The Lemaitre-Tolman-Bondi (LTB) solutions with a dust source [1],[2], [3] are widely popular models of cosmological inhomogeneities (see [2] for a comprehensive review). We present a generalization in which the LTB metric is kept but the source is replaced by an imperfect fluid with anisotropic pressure, under the assumption that matter-energy density is decomposed as $\rho = \rho^{(m)} + \rho^{(r)}$, a mixture of "rest mass" and "radiation" components (a mixture of non-relativistic and relativistic matter), so that $\rho^{(r)} = 3p$, where p is the isotropic pressure. The purpose of this paper is to derive the important geometric properties of the solutions within a convenient framework and then to examine the compatibility of the models with the physics of radiation-matter sources.

The study of inhomogeneous cosmological models such the LTB models is a well motivated and justified endeavor. First of all, it complements the usual perturbative approach by allowing one to study the non-linear evolution of inhomogeneities. Also, a nearly isotropic Cosmic Microwave Background Radiation (CMB) does not rule out an inhomogeneous universe compatible with current CMB observations [4],[5]. Moreover, the models we examine here are of particular interest when it is necessary to consider the radiation component and the dissipative processes associated with its interaction with non-relativistic matter [6], [7], [8]. On the other hand, even if it is reasonable to use a dust source model as a theoretical matter model for present day universe, this source cannot describe the CMB and does not allow one to deal with temperatures of the photon gas.

Various physical interpretations and observational bounds have been proposed for anisotropic stresses in cosmic matter sources[10],[11]. If the mixture components are not interacting, then we have a mixture of collision-less non-relativistic matter and CMB with anisotropic pressure understood as the quadrupole moment of the distribution function for the photon gas[12] - [18] under a Kinetic Theory approach[19],[20]. For a matter-radiation mixture in which there is interaction between the components (a radiative photon-electron interaction), the anisotropic pressure can be interpreted

as shear viscosity within a causal irreversible thermodynamic approach [9], [21] -[22]. This provides an adequate model for the radiative era of cosmic evolution, from after nucleosynthesis to up to "matter" and "radiation" decoupling.

In section II, Einstein's field equations are integrated up to a Friedmann-like equation, a quadrature in which the free parameters are three initial value functions related to the average of initial energy densities $\rho_i^{(m)}$, $\rho_i^{(r)}$ and ${}^{(3)}R$, the 3-dimensional Ricci scalar along the initial surface $t = t_i$. In section III, we describe the possible physical interpretations of the models source focusing on the interacting matter-radiation mixture. In sections IV and V we express the free parameters in terms of suitable volume averages and initial contrast functions defined along $t = t_i$, leading to simplified and elegant forms for all the relevant geometric and physical quantities. These quantities become fully determined (up to initial conditions) once the Friedmann-like quadrature is integrated (appendix A) yielding canonical and parametric solutions. The models contain three important particular cases presented in appendix B. In sections VI and VII, we examine the state variables of the model under small density contrasts. Theses small contrasts lead to two important classes of initial conditions equivalent to the definition of adiabatic and quasi-adiabatic perturbations of the initial densities and the initial curvature. In section VIII, we study the thermodynamic consistency of the coupled mixture model.

Different variants of the imperfect fluid generalization of LTB solutions have been presented previously: using the equation of state of a non-relativistic ideal gas [14], considering various possible ideal gas equations of state and a generalization to non-spherical geometry of the Szekeres-Szafron type [15] and the parabolic case for the matter-radiation mixture [16]. The present paper extend and complement the results of previous literature with respect to the specific cases of spherically symmetric and curved (elliptic and hyperbolic) models with a radiation-matter mixture source.

II. FIELD EQUATIONS.

Consider the usual LTB metric

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \frac{Y'^2}{1-K}\,\mathrm{d}r^2 + Y^2\left[\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2\right], \tag{1}$$

where $Y = Y(t,r)$, $K = K(r)$ and a prime denotes derivative with respect to r. Instead of the usual dust source, we shall consider the stress-energy tensor of a fluid with anisotropic pressure

$$T^{ab} = \rho u^a u^b + p h^{ab} + \Pi^{ab}, \quad h^{ab} = c^{-2}u^a u^b + g^{ab}, \quad \Pi^a{}_a = 0\,, \tag{2}$$

where the most general form for the anisotropic pressure tensor for the metric (1) with matter source (2) is given by: $\Pi^a{}_b = \mathrm{diag}\,[0, -2P, P, P]$, with $P = P(t,r)$ to be determined by the field equations. Einstein's field equations for (1) and (2) are

$$\frac{8\pi G}{c^4}\rho \;=\; -G^t{}_t \;=\; \frac{\left[Y(\dot{Y}^2+K)\right]'}{Y^2Y'}, \tag{3}$$

$$\frac{8\pi G}{c^4}p \;=\; \frac{1}{3}(2G^\theta{}_\theta + G^r{}_r) \;=\; -\frac{\left[Y(\dot{Y}^2+K)+2Y^2\ddot{Y}\right]'}{3Y^2Y'}, \tag{4}$$

$$\frac{8\pi G}{c^4}P \;=\; \frac{1}{3}(G^\theta{}_\theta - G^r{}_r) \;=\; -\frac{Y}{6Y'}\left[\frac{Y(\dot{Y}^2+K)+2Y^2\ddot{Y}}{Y^3}\right]', \qquad (5)$$

where $\dot{Y} = u^a Y_{,a} = Y_{,ct}$. In order to integrate these equations we need to impose a relation between ρ and p but no previous assumption on P is necessary.

Consider ρ in (2) to be the sum of two contributions: non-relativistic matter described by dust plus radiation energy density ($\rho^{(r)}$)

$$\rho \;=\; \rho^{(m)} \;+\; \rho^{(r)}, \quad p \;=\; \frac{1}{3}\rho^{(r)}, \qquad \text{with:}\;\; \frac{8\pi G}{c^4}\,\rho^{(m)} \;=\; \frac{2M'}{Y^2Y'}, \qquad (6)$$

where $M = M(r)$, while the form of $\rho^{(r)}$ will be discussed further ahead. Inserting (6) into (3) and (4) and integrating once with respect to t leads to the Friedmann equation

$$\dot{Y}^2 \;=\; \frac{1}{Y}\left[2M \;+\; W\frac{Y_i}{Y}\right] \;-\; K, \qquad (7)$$

where $W = W(r)$ and $Y_i = Y(t_i, r)$ for an arbitrary fixed value $t = t_i$. The interpretation for M and W follows by substituting (7) into equations (3), (4) and (5)

$$\begin{aligned}
\frac{8\pi G}{c^4}\,\rho\, Y^2Y' \;&=\; \left[2M + W\frac{Y_i}{Y}\right]',\\
\frac{8\pi G}{c^4}\,p\, Y^2Y' \;&=\; \frac{1}{3}\left[W\frac{Y_i}{Y}\right]',\\
\frac{8\pi G}{c^4}\,P\frac{Y'}{Y} \;&=\; -\frac{1}{6}\left[\frac{WY_i}{Y^4}\right]'. \qquad (8)
\end{aligned}$$

Since Y^2Y' is proportional to the determinant of the spatial part of the metric (1), it is a covariant measure of proper local volumes. Therefore M' and W' must have units of length and so it is convenient to define them in terms of the initial energy densities of matter and radiation

$$2M \;=\; \frac{8\pi G}{c^4}\int \rho_i^{(m)} Y_i^2 Y_i' dr, \quad W \;=\; \frac{8\pi G}{c^4}\int \rho_i^{(r)} Y_i^2 Y_i' dr, \quad \rho_i^{(r)} \;=\; 3p_i, \qquad (9)$$

where $\rho_i^{(m)}$, $\rho_i^{(r)}$ are $\rho^{(m)}$, $\rho^{(r)}$ evaluated at $t = t_i$.

For a non-rotating fluid with a geodesic 4-velocity the two nonzero kinematic parameters are: the expansion scalar, $\Theta \equiv u^a{}_{;a}$, and the shear tensor, $\sigma_{ab} = u_{(a;b)} - (\Theta/3)h_{ab}$. These parameters for (1) take the form

$$\Theta \;=\; \frac{2\dot{Y}}{Y} \;+\; \frac{\dot{Y}'}{Y'}, \qquad (10)$$

$$\sigma^a{}_b \;=\; \text{diag}\,[0, -2\sigma, \sigma, \sigma], \quad \sigma \;\equiv\; \frac{1}{3}\left(\frac{\dot{Y}}{Y} - \frac{\dot{Y}'}{Y'}\right), \quad \sigma_{ab}\sigma^{ab} \;=\; 6\sigma^2, \qquad (11)$$

Under the assumptions (6) the energy and momentum balance law, $T^{ab}{}_{;b} = 0$, associated with (1) and (2) gives

$$\dot{\rho}^{(r)} \;+\; \frac{4}{3}\,\Theta\,\rho^{(r)} \;+\; 6\,\sigma\,P \;=\; 0,$$

$$h_a^b\,(\,p_{,b} + \Pi^c{}_{b;c}\,) \;=\; 0 \quad\Rightarrow\quad (\,p - 2P\,)' - 6P\frac{Y'}{Y} \;=\; 0,$$

clearly illustrating how the pressure gradient is exactly balanced by the divergence of $\Pi^a{}_b$, allowing for non-zero pressure gradients to be compatible with geodesic motion of comoving matter.

III. MATTER-RADIATION MODELS.

The relation between ρ and p given by (6) suggests that the matter source can be understood as a mixture of non-relativistic matter (to be referred as "matter") and ultra-relativistic matter (to be referred as "radiation"), both characterized by the same 4-velocity. Therefore, a suitable physical interpretation for this source is an interactive, 'tightly coupled' mixture of matter and radiation examined under a hydrodynamical approach. The anisotropic pressure becomes a dissipative term (a shear viscosity) that must be examined within a thermodynamic framework[21] -[22], [23],[24]. This type of source provides a convenient description for the "radiative era", after nucleosynthesis and before decoupling of radiation and matter. An alternative approach is that of a decoupled mixture (non-relativistic matter plus the CMB) under the framework of Kinetic Theory [12] - [20]. In this paper we shall consider only the hydrodynamical approach applicable to the radiative era, leaving the study of a decoupled mixture for a future work.

The 'tightly coupled' mixture of non-relativistic matter and radiation (a photon gas) is characterized by local thermal equilibrium (common temperature) among the components. This situation implies that interaction timescales are smaller than the cosmic expansion timescale, thus a hydrodynamical approach is valid so that this interactive mixture behaves as a single dissipative fluid. Since heat flux necessarily vanishes for the LTB metric (1), this fluid can be described by the momentum-energy tensor (2) where Π^{ab} now becomes the shear viscous tensor. Considering that non-relativistic matter can be modeled by a classical monatomic ideal gas, while the photon gas satisfies the Stefan-Boltzmann law, the tightly coupled mixture of these components requires that ρ and p in (2) must comply with the equation of state

$$\rho \;=\; m\,c^2\,n \;+\; \frac{3}{2}\,n\,k\,T \;+\; a\,T^4, \quad p \;=\; n\,k\,T \;+\; \frac{1}{3}\,a\,T^4, \tag{12}$$

where k, a are Boltzmann and Stephan-Boltzmann constants, T is the common mixture temperature, m is the mass of the most representative species of non-relativistic particles and n is particle number density, satisfying (we assume there is no net creation or annihilation of particles) the conservation law

$$(n\,u^a)_{;a} \;=\; 0 \quad \Rightarrow \quad n \;=\; \frac{N(r)}{Y^2 Y'}, \tag{13}$$

where $N(r)$ is an arbitrary function. If $nkT \ll aT^4$, or equivalently $aT^3/nk \gg 1$, but mc^2n/aT^4 is not negligible, then (12) can be approximated by[9],[21]

$$\rho \;=\; m\,c^2\,n + a\,T^4, \quad p \;=\; \frac{1}{3}\,a\,T^4, \tag{14}$$

Comparing the conservation law (13) and the equation of state (14) with (6a) and (6b), it is evident that the generalized LTB solutions can provide a model for the radiation-matter tightly coupled mixture if we identify

$$\rho^{(m)} \;=\; m\,c^2\,n, \quad m\,c^2\,N \;=\; 2\,M' \;=\; \frac{8\pi G}{c^4}\,\rho_i^{(m)}\,Y_i^2 Y_i', \quad \rho_i^{(m)} \;=\; m\,c^2\,n_i, \tag{15}$$

so that the radiation energy density and the temperature are given by

$$\frac{[W\,Y_i/Y]'}{Y^2 Y'} \;=\; \frac{8\pi G}{c^4}\,\rho^{(r)} \;=\; \frac{8\pi G}{c^4}\,a\,T^4, \quad \frac{8\pi G}{c^4}\,a\,T_i^4 \;=\; W'\,Y_i^2 Y_i'. \tag{16}$$

As an alternative to (12) and (14), it is possible to consider, instead of the Stefan-Boltzmann law, $\rho^{(r)} = aT^4$, the energy density and pressure of the radiation component as those of an ideal ultra-relativistic gas

$$p^{(r)} \;=\; n^{(r)}\,k\,T, \quad \rho^{(r)} \;=\; 3p^{(r)}, \quad n^{(r)} \;=\; \frac{N^{(r)}}{Y^2 Y'} \tag{17}$$

where $n^{(r)}$ is the corresponding particle number density, independently satisfying a conservation law like (13) with $N^{(r)} = N^{(r)}(r)$. This approach has been followed previously in [15] and [16], the advantage being a simpler expression than (16) for the temperature in terms of the gradients of Y

$$\frac{8\pi G}{c^4}\,p\,Y^2 Y' \;=\; \frac{1}{3}\left[W\frac{Y_i}{Y}\right]' \;=\; N^{(r)}\,k\,T, \tag{18}$$

In global thermal equilibrium, the fact that photon entropy per barion is conserved: $s^{(e)} = (4/3)\,a\,T^3/n =$ const. implies that the ratio of photons to baryons is constant and $a\,T^4 \propto n\,k\,T$, hence the Stefan-Boltzmann and ideal gas laws are equivalent. Since we shall consider near equilibrium conditions in which these two laws should be almost (but not exactly) equivalent, then $a\,T^3/(n\,k)$ is proportional to the (approximately constant) number of photons per non-relativistic particle. If the latter are baryons and electrons, then this quantity is very large, thus justifying the approximation leading from (12) to (14). If temperatures are high enough so that the ratio $aT^4/(mc^2 n)$ is low enough for creation/annihilation processes to cancel each other (hence (13) holds), then (14) provides a reasonable description of cosmic matter in the radiative era. The dominant radiative processes characteristic of this era mostly involve the photon-electron interaction (Thomson and Compton scattering, Brehmstrallung, etc) [6], [21] and [22]. The temperature range for the radiative era (up to matter-radiation decoupling) is roughly $4 \times 10^3 K. < T < 10^6 K.$ The anisotropic pressure $\Pi^a{}_b$ is now the shear viscosity tensor, hence its evolution law must be consistent with the shear viscosity transport equation of a causal irreversible thermodynamic theory. In the particular case when shear viscosity is the single dissipative flux, the entropy per particle and shear viscosity transport equation provided by Extended Irreversible Thermodynamics are

$$s \;=\; s^{(e)} \;+\; \frac{\alpha}{nT}\,\Pi_{ab}\Pi^{ab}, \quad \Rightarrow \quad (snu^a)_{;a} \;=\; \dot{s}nu^a \;\geq\; 0, \tag{19}$$

$$\tau\,\dot{\Pi}_{cd}\,h_a^c h_b^d \;+\; \Pi_{ab}\left[1 + \frac{1}{2}T\eta\left(\frac{\tau}{T\eta}\,u^c\right)_{;c}\right] \;+\; 2\,\eta\,\sigma_{ab} \;=\; 0, \tag{20}$$

where $s^{(e)} = (4/3)aT^3/n$ is the equilibrium entropy per particle, taken (approximately) as the initial photon entropy per non-relativistic particle, η, τ are the coefficient of shear viscosity and a relaxation time which, together with α, are phenomenological coefficients whose functional form depends on the fluid under consideration. A "truncated" version of (20), also known as the "Maxwell-Cattaneo" transport equation, is often used[22], [23] for the sake of mathematical simplicity:

$$\tau\,\dot{\Pi}_{cd}\,h_a^c h_b^d \;+\; \Pi_{ab} \;+\; 2\,\eta\,\sigma_{ab} \;=\; 0, \tag{21}$$

Although this truncated equation satisfies causality and stability requirements, numerical and theoretical studies indicate that it might be problematic[25]. In spite of the obvious limitation of having only one dissipative flux, the generalized LTB models provide arguments to infer the ranges of applicability of the truncated and full equations.

The application of the thermodynamic formalism to the generalized LTB models, considered as a models of a hydrodynamical 'tightly coupled' mixture of matter and radiation, requires a convenient selection of the phenomenological quantities η, α, τ, and then solving the appropriate transport equation (either (20) or (21)). In order to proceed with this task we will find in the following sections an intuitive characterization of initial conditions and suitable forms for the state variables.

IV. INITIAL CONDITIONS AND VOLUME AVERAGES

It is convenient to transform the Friedmann equation (7) into the simpler quadrature

$$\dot{y}^2 \;=\; \frac{-\kappa\, y^2 + 2\mu\, y + \omega}{y^2}, \tag{22}$$

where

$$y \;=\; \frac{Y}{Y_i}, \quad \mu \;\equiv\; \frac{M}{Y_i^3}, \quad \omega \;\equiv\; \frac{W}{Y_i^3}, \quad \kappa \;\equiv\; \frac{K}{Y_i^2}, \quad Y_i \;=\; Y(t_i, r)$$

Bearing in mind (9), assuming that the initial hypersurface $t = t_i$ is everywhere regular and contains at least a symmetry center [16] [15], we can express the quantities μ, ω, κ as the following type of volume averages along $t = t_i$

$$\langle \rho_i^{(m)} \rangle \;\equiv\; \frac{\int \rho_i^{(m)} Y_i^2 Y_i' dr}{\int Y_i^2 Y_i' dr} \quad \Rightarrow \quad 2\,\mu \;=\; \frac{8\pi G}{3c^4}\,\langle \rho_i^{(m)} \rangle,$$

$$\langle \rho_i^{(r)} \rangle \;\equiv\; \frac{\int \rho_i^{(r)} Y_i^2 Y_i' dr}{\int Y_i^2 Y_i' dr} \quad \Rightarrow \quad \omega \;=\; \frac{8\pi G}{3c^4}\,\langle \rho_i^{(r)} \rangle,$$

$$\langle {}^{(3)}\mathcal{R}_i \rangle \;\equiv\; \frac{\int {}^{(3)}\mathcal{R}_i Y_i^2 Y_i' dr}{\int Y_i^2 Y_i' dr} \quad \Rightarrow \quad \kappa \;=\; \frac{1}{6}\,\langle {}^{(3)}\mathcal{R}_i \rangle,, \tag{23}$$

where

$${}^{(3)}\mathcal{R}_i \;=\; \frac{2(K\,Y_i)'}{Y_i^2 Y_i'},$$

is the Ricci scalar of (1) evaluated at $t = t_i$ and the range of integration goes from the symmetry center up to a comoving sphere marked by r. Notice that these averages (save for $K = 0$) do not coincide with the local proper volume defined by (1), namely: $Y_i^2 Y_i'/\sqrt{1-K} \;=\; (1/3)d(Y_i^3)/\sqrt{1-K}$, though they can also be characterized covariantly, since the metric function Y (and so Y_i) is, in spherical symmetry, the "curvature radius", or the proper radius of the orbits of the rotation group SO(3). The volume averages (23) lead to a compact and elegant mathematical description of initial conditions. From the volume averages in (23) we can define the functions $\Delta_i^{(m)}$, $\Delta_i^{(r)}$, $\Delta_i^{(k)}$ satisfying the following appealing relations

$$\begin{aligned} \rho_i^{(m)} &= \langle \rho_i^{(m)} \rangle \left[1 + \Delta_i^{(m)}\right], \\ \rho_i^{(r)} &= \langle \rho_i^{(r)} \rangle \left[1 + \Delta_i^{(r)}\right], \\ {}^{(3)}\mathcal{R}_i &= \langle {}^{(3)}\mathcal{R}_i \rangle \left[1 + \Delta_i^{(k)}\right], \end{aligned} \tag{24}$$

justifying their interpretation as initial density and curvature "contrast functions", as they provide a measure of the contrast of initial value functions $\rho_i^{(m)}$, $\rho_i^{(r)}$, ${}^{(3)}\mathcal{R}_i$ with respect to their volume averages along the initial hypersurface $t = t_i$. Because of their definition in terms of the volume averages (23), the value of the contrast functions at a given $r = r_*$ depend on the values of the functions $\rho_i^{(m)}$, $\rho_i^{(r)}$, ${}^{(3)}\mathcal{R}_i$ on the integration range $0 \le r \le r_*$, in which we assume that $r = 0$ marks a symmetry center characterized by $Y(t,0) = \dot{Y}(t,0) = 0$ [16] [15]. Therefore, for a rest mass density lump $\rho_i^{(m)}$ decreases with increasing r, while for a rest mass density void it increases with increasing r, hence (from (24)) we have $-1 \le \Delta_i^{(m)} \le 0$ for a lump and $\Delta_i^{(m)} \ge 0$ for a void. The same criterion holds for the initial radiation density and scalar curvature, $\rho_i^{(r)}$ and ${}^{(3)}\mathcal{R}_i$.

From (23) and (24), it is straightforward to verify that the functions $\rho_i^{(m)}, \rho_i^{(r)}, {}^{(3)}\mathcal{R}_i$ relate to $\Delta_i^{(m)}$, $\Delta_i^{(r)}$, $\Delta_i^{(k)}$ by

$$\epsilon_1 = \frac{\omega}{\mu} = \frac{2\rho_i^{(r)}}{\rho_i^{(m)}}\,\frac{1+\Delta_i^{(m)}}{1+\Delta_i^{(r)}}, \quad \epsilon_2 = \frac{\kappa}{\mu} = \frac{c^4\,{}^{(3)}\mathcal{R}_i}{8\pi G\,\rho_i^{(m)}}\,\frac{1+\Delta_i^{(m)}}{1+\Delta_i^{(k)}}, \tag{25}$$

and

$$\Delta_i^{(r)} = \frac{\omega'/\omega}{Y_i'/Y_i}, \quad \Delta_i^{(m)} = \frac{\mu'/\mu}{Y_i'/Y_i}, \quad \Delta_i^{(k)} = \frac{\kappa'/\kappa}{Y_i'/Y_i}. \tag{26}$$

Since the state variables $\rho^{(m)}$, $\rho^{(r)}$, P and kinematic parameters Θ, σ are given in terms of quantities that will depend on the initial value functions μ, ω, κ and their gradients, it is useful to be able to express these gradients, by means of (25) and (26), in terms of the more intuitively appealing initial contrast functions defined by (24). We examine this point in the following section.

V. STATE VARIABLES AND KINEMATIC PARAMETERS

Using (8) (15) and (16), we can rewrite the state variables $\rho^{(m)}$, $\rho^{(r)}$, P in a more appealing form

$$\rho^{(m)} = m\,c^2\,n = \frac{\rho_i^{(m)}}{y^3\Gamma}, \quad \rho_i^{(m)} = m\,c^2\,n_i \tag{27}$$

$$\begin{aligned} \rho^{(r)} = 3\,p = \frac{\rho_i^{(r)}\Psi}{y^4\Gamma}, \quad \rho_i^{(r)} &= a\,T_i^4 \quad \text{(Stefan-Boltzmann)},\\ \rho_i^{(r)} &= 3n_i^{(r)}\,k\,T_i \quad \text{(Ideal Gas)}, \end{aligned} \tag{28}$$

$$T = \frac{T_i}{y}\left(\frac{\Psi}{\Gamma}\right)^{1/4} \text{(Stefan-Boltzmann)}, \; T = \frac{T_i\,\Psi}{y} \text{(Ideal Gas)}, \tag{29}$$

$$P = \frac{\rho_i^{(r)}\Phi}{6y^4\Gamma}, \tag{30}$$

where the auxiliary functions Γ, Ψ, Φ, characterizing the spacial dependence of ρ, p, P, are given by

$$\Gamma = \frac{Y'/Y}{Y_i'/Y_i} = 1 + \frac{y'/y}{Y_i'/Y_i}, \quad \Psi = \frac{4+3\Delta_i^{(r)} - \Gamma}{3[1+\Delta_i^{(r)}]}, \quad \Phi = \frac{4+3\Delta_i^{(r)} - 4\Gamma}{3[1+\Delta_i^{(r)}]}, \tag{31}$$

where $\Delta_i^{(r)}$ has been defined in (24). The function Γ can be obtained as a function of y, μ, ω, κ and the initial contrast functions by performing the integral quadrature of (22). Let

$$Z \;\equiv\; \int \frac{y dy}{\left[-\kappa y^2 + 2\mu y + \omega\right]^{1/2}} \;=\; \frac{1}{\sqrt{\mu}} \int \frac{y dy}{\left[-\epsilon_2\, y^2 + 2\, y + \epsilon_1\right]^{1/2}}, \tag{32}$$

where

$$Z_i(\mu,\omega,f) \;=\; Z|_{y=1}, \quad \epsilon_1 \;\equiv\; \frac{\omega}{\mu}, \quad \epsilon_2 \;\equiv\; \frac{\kappa}{\mu},$$

then, by using the chain rule on $Z - Z_i$ we can obtain for Γ, Ψ and Φ in (31) the following expressions

$$\Gamma \;=\; 1 \;-\; 3\,A\,\Delta_i^{(m)} \;-\; 3\,B\,\Delta_i^{(r)} \;-\; 3\,C\,\Delta_i^{(k)}, \tag{33}$$

$$\Psi \;=\; \frac{1 \;+\; A\,\Delta_i^{(m)} \;+\; (1+B)\,\Delta_i^{(r)} \;+\; C\,\Delta_i^{(k)}}{1 \;+\; \Delta_i^{(r)}}, \tag{34}$$

$$\Phi \;=\; \frac{4\,A\,\Delta_i^{(m)} \;+\; (1+4B)\,\Delta_i^{(r)} \;+\; 4\,C\,\Delta_i^{(k)}}{1 \;+\; \Delta_i^{(r)}}, \tag{35}$$

where

$$A = \frac{\mu\,[\partial(Z-Z_i)/\partial\mu]}{y\,[\partial Z/\partial y]},\; B = \frac{\omega\,[\partial(Z-Z_i)/\partial\omega]}{y\,[\partial Z/\partial y]},\; C = \frac{\kappa\,[\partial(Z-Z_i)/\partial\kappa]}{y\,[\partial Z/\partial y]}, \tag{36}$$

In order to obtain the explicit functional forms for A, B, C above we need to evaluate Z explicitly. This is done in Appendix A, while a convenient interpretation as "initial contrast functions" has been provided in the previous section for the functions $\Delta_i^{(m)}$, $\Delta_i^{(r)}$, $\Delta_i^{(k)}$. It is important to mention that no *a priori* assumption on P was made in order to obtain (30). This specific form of anisotropic pressure (shear viscosity) follows directly from (3), (4), (14) and (22).

In terms of the new variables, the kinematic parameters, Θ, σ, introduced in (10) and (11), take the form

$$\begin{aligned}\Theta \;&=\; \frac{3\dot{y}}{y} \;+\; \frac{\dot{\Gamma}}{\Gamma}\\ &=\; \frac{\sqrt{Q}}{y}\,\frac{1-(3A+yA_{,y})\Delta_i^{(m)}-(3B+yB_{,y})\Delta_i^{(r)}-(3C+yC_{,y})\Delta_i^{(k)}}{1-3A\Delta_i^{(m)}-3B\Delta_i^{(r)}-3C\Delta_i^{(k)}}\end{aligned} \tag{37}$$

$$\sigma \;=\; -\frac{\dot{\Gamma}}{3\Gamma} \;=\; \frac{\sqrt{Q}}{y}\;\frac{A_{,y}\Delta_i^{(m)}+B_{,y}\Delta_i^{(r)}+C_{,y}\Delta_i^{(k)}}{1-3A\Delta_i^{(m)}-3B\Delta_i^{(r)}-3C\Delta_i^{(k)}}, \tag{38}$$

where $Q \equiv -\kappa y^2 + 2\mu y + \omega$ and a sub-index $_y$ means partial derivative with respect to y. The new solutions presented so far become fully determined once the Friedmann equation (22) is integrated (Z is explicitly known) for specific initial conditions provided by μ, ω and κ. This integration is presented in appendix A and particular cases of interest are presented in appendix B. The conditions for the models to comply with regularity at the center and energy conditions are given in [16] and [15].

VI. SMALL DENSITY CONTRASTS AND AN ADIABATIC EVOLUTION.

The thermodynamic study of the generalized LTB models, under the framework of Extended Irreversible Thermodynamics, necessarily assumes small deviations from equilibrium. Since the equilibrium limit of the models (excluding the dust limit which has trivial thermodynamics) is the FLRW limit that follows by setting the initial contrast functions to zero, then a "near equilibrium" evolution can be related to a "near FLRW" evolution that follows by assuming "near homogeneous initial conditions", or in other words, small initial contrast functions, *ie*:

$$|\Delta_i^{(m)}| \ll 1, \quad |\Delta_i^{(r)}| \ll 1, \quad |\Delta_i^{(k)}| \ll 1, \tag{39}$$

implying

$$\rho_i^{(m)} \approx \langle \rho_i^{(m)} \rangle, \quad \rho_i^{(r)} \approx \langle \rho_i^{(r)} \rangle, \quad {}^{(3)}\mathcal{R}_i \approx \langle {}^{(3)}\mathcal{R}_i \rangle,$$

$$2\mu \approx \frac{8\pi G}{3c^4}\,\rho_i^{(m)}, \quad \omega \approx \frac{8\pi G}{3c^4}\,\rho_i^{(r)}, \quad \kappa \approx \frac{1}{6}\,{}^{(3)}\mathcal{R}_i,$$

$$\epsilon_1 \approx \frac{2\rho_i^{(r)}}{\rho_i^{(m)}}, \quad \epsilon_2 \approx \frac{c^4}{8\pi G}\,\frac{{}^{(3)}\mathcal{R}_i}{\rho_i^{(m)}}, \qquad (\epsilon_1, \epsilon_2 \approx \text{const.}), \tag{40}$$

where the exact form of ϵ_1, ϵ_2 is given by (25). Under these assumptions, all functions depending on (μ, ω, κ, y) become approximately functions of y only. From (32), (33), (34), (35) and (36), this implies

$$Z \approx Z(y),\ t - t_i \approx Z(y) - Z_i,\ A \approx A(y),\ B \approx B(y),\ C \approx C(y), \tag{41}$$

with $Z_i \approx$ const., so that hypersurfaces $y =$ const. approximate hypersurfaces $t =$ const. and y becomes the time parameter. The radial dependence in quantities like Γ, Ψ, Φ is then contained in the initial contrast functions. Regarding the state variables, a Taylor series expansion around $\Delta_i^{(m)}$, $\Delta_i^{(r)}$, $\Delta_i^{(k)}$ yields at first order

$$\frac{1}{\Gamma} \approx 1 + \delta, \quad \frac{\Psi}{\Gamma} \approx 1 + \frac{4}{3}\,\delta, \quad \frac{\Phi}{\Gamma} \approx \Delta_i^{(r)} + \frac{4}{3}\,\delta, \tag{42}$$

where we can formally identify the "perturbation" as

$$\delta \equiv 3\,[A\,\Delta_i^{(m)} + B\,\Delta_i^{(r)} + C\,\Delta_i^{(k)}] = 1 - \Gamma, \tag{43}$$

so that, from (27) and (42), we have

$$\rho^{(m)} \approx \frac{\rho_i^{(m)}}{y^3}\,[1+\delta]\,,\ \rho^{(r)} \approx \frac{\rho_i^{(r)}}{y^4}\left[1 + \frac{4}{3}\,\delta\right],\ P \approx \frac{\rho_i^{(r)}}{6y^4}\left[\Delta_i^{(r)} + \frac{4}{3}\,\delta\right], \tag{44}$$

Therefore, under the "small contrast approximation" given by (39), (40) and (41) the models follow a nearly homogeneous evolution with respect to a scale factor $y \approx y(t)$, deviating from homogeneity in a way (as given by equations (44)) that is formally analogous to that of adiabatic perturbations. This situation holds for whatever form of the contrast functions, as long as these adimensional quantities are very small (so that the initial functions $\rho_i^{(m)}, \rho_i^{(r)}, {}^{(3)}R_i$ are almost constant). Notice that, at first order expansion on the contrast functions, we have for (29)

$$\Psi \approx \left[\frac{\Psi}{\Gamma}\right]^{1/4} \approx 1 + \frac{1}{3}\,\delta \qquad \Rightarrow \qquad T \approx \frac{T_i}{y}\left[1 + \frac{1}{3}\,\delta\right], \tag{45}$$

so that both, the Stefan-Boltzmann and Ideal Gas laws, yield the same expression for T. Therefore, as long as we assume the small contrasts approximation, the Stefan-Boltzmann and Ideal Gas laws yield the same result and we can use them indistinctly.

The resemblance of (44) to expressions characteristic of adiabatic perturbations is not surprising: the generalized LTB models under consideration (under a thermodynamic approach) have zero heat flux and so can be associated with thermodynamic processes that are *indeed* adiabatic, though irreversible (because of the shear viscosity). The connection between the contrast functions and adiabatic perturbations on a radiation-matter 'tight coupling' is an interesting feature that deserves proper examination and will be studied in a separate paper. We will use this analogy as a theoretical tool for understanding the type of quasi-homogeneous evolution and for defining suitable initial conditions for the study of the thermodynamic consistency of the models.

VII. ADIABATIC AND QUASI-ADIABATIC INITIAL CONDITIONS.

As shown in [16], the quantity $\Delta_i^{(s)} = \Delta_i^{(r)} - (4/3)\Delta_i^{(m)}$ becomes, under the assumptions (39), (40) and (41), a sort of average change of photon entropy per barion at the initial hypersurface. Hence, it is convenient to rephrase initial conditions in terms of this quantity. After some algebraic manipulation on the explicit general forms given in Appendix A for the functions A, B, C defined by (36), we find that the following restrictions on the initial contrast functions

$$\frac{4}{3}\Delta_i^{(m)} = \Delta_i^{(r)} - \Delta_i^{(s)}, \quad \Delta_i^{(k)} = \frac{1}{2}\Delta_i^{(r)}, \tag{46}$$

leads to the following compact forms for the function Γ defined in (33)

$$\begin{aligned}
\Gamma &= 1 + \frac{3}{4}\left[1 - \frac{\sqrt{q}}{y^2\sqrt{q_i}}\right]\Delta_i^{(r)} + \\
&\quad \frac{3}{4y^2}\left[8\epsilon_1^2 + 4\epsilon_1 y - y^2 - \frac{\sqrt{q}}{\sqrt{q_i}}\left(8\epsilon_1^2 + 4\epsilon_1 - 1\right)\right]\Delta_i^{(s)}, \\
&\quad \text{for the parabolic case} \quad (\epsilon_2 = 0), \\
\Gamma &= 1 + \frac{3}{4}\left[1 - \frac{\sqrt{q}}{y^2\sqrt{q_i}}\right]\Delta_i^{(r)} + \frac{9}{4\epsilon_2\lambda_0 y^2} \\
&\quad \left[\epsilon_1\left(1 - \frac{\sqrt{q}}{y^2\sqrt{q_i}}\right) - (\lambda_0 \pm 1)\left(y - \frac{\sqrt{q}}{\sqrt{q_i}}\right) \pm \frac{\lambda_0(\eta - \eta_i)\sqrt{q}}{\sqrt{|\epsilon_2|}}\right]\Delta_i^{(s)}, \\
&\quad \text{elliptic } (\epsilon_2 > 0, +\text{sign}) \text{ and hyperbolic } (\epsilon_2 < 0, -\text{ sign}) \text{ cases,}
\end{aligned} \tag{47}$$

where $\epsilon_1 = \omega/\mu$, $\epsilon_2 = \kappa/\mu$ where defined in (25) and

$$q = -\epsilon_2\, y^2 + 2\, y + \epsilon_1, \quad q_i = -\epsilon_2 + 2 + \epsilon_1,$$

$$\lambda_0 = 1 \pm \epsilon_1|\epsilon_2|, \quad \text{elliptic}\,(+\text{ sign}), \quad \text{hyperbolic}\,(-\text{ sign}),$$

$$\eta = \arccos\left(\frac{1 - \epsilon_2\, y}{\sqrt{\lambda_0}}\right), \quad \eta_i = \arccos\left(\frac{1 - \epsilon_2}{\sqrt{\lambda_0}}\right), \quad \text{elliptic case,}$$

$$\eta = \text{arccosh}\left(\frac{1 + |\epsilon_2|\, y}{\sqrt{\lambda_0}}\right), \ \eta_i = \text{arccosh}\left(\frac{1 + |\epsilon_2|}{\sqrt{\lambda_0}}\right), \ \text{hyperbolic case,}$$

It is evident, by looking at these forms for Γ above, that this function takes a very simple form

$$\Gamma \;=\; 1 \;+\; \frac{3}{4}\left[1-\frac{\sqrt{q}}{y^2\sqrt{q_i}}\right]\,\Delta_i^{(r)}, \tag{48}$$

valid for all cases (parabolic, elliptic and hyperbolic) if we use initial conditions given by

$$\Delta_i^{(s)}=0 \quad\Rightarrow\quad \Delta_i^{(m)}=\frac{3}{4}\Delta_i^{(r)},\quad \Delta_i^{(k)}=\frac{1}{2}\Delta_i^{(r)}, \tag{49}$$

Since all state variables, kinematic parameters and auxiliary functions are constructed from Γ, y and initial value functions, the assumption (49) leads to very simplified forms for all expressions. Considering the initial contrast functions $\Delta_i^{(m)},\Delta_i^{(r)},\Delta_i^{(k)}$ in (24) as formally analogous to exact perturbations on $\rho_i^{(m)},\rho_i^{(r)},{}^{(3)}\mathcal{R}_i$, the factor 4/3 relating $\Delta_i^{(m)}$ and $\Delta_i^{(r)}$ in (49) is reminiscent of adiabatic perturbations on the initial value functions. Hence, following [16] and [26], we will denote (49) as "adiabatic initial conditions", while the more general case (46) with $\Delta_i^{(s)}\neq 0$ will be refered to as "quasi-adiabatic initial conditions". For the remaining of this paper we will only consider initial conditions of either these two types, under the "small contrast approximation" given by (39), (40) and (41).

VIII. THERMODYNAMIC CONSISTENCY

In order for the models presented in this paper to be physically meaningful they must satisfy energy conditions and must be compatible with causal Extended Irreversible Thermodynamics. Before to start this analysis we need first to provide an expression for the phenomenological quantities η, α, τ (coefficient of shear viscosity, relaxation time). Following the approach used in [16] and [26], we consider for a matter-radiation mixture interacting via radiative processes, the "radiative gas" model, associated with the photon-electron interaction, which provides the following forms for η and α [21] -[22]

$$\eta \;=\; \frac{4}{5}\,p\,\tau \;=\; \frac{4}{15}\,a\,T^4\tau,\quad \alpha \;=\; -\frac{\tau}{2\eta} \;=\; -\frac{15}{8\,a\,T^4}, \tag{50}$$

Inserting (50) into (19) and (20), the latter equations become

$$\dot{s} \;=\; \frac{5\,k}{8\,\tau}\,\frac{\Pi_{ab}\Pi^{ab}}{p^2} \;=\; \frac{15\,k}{4\,\tau}\,\frac{P^2}{p^2}, \tag{51}$$

$$\dot{P}+\left(\frac{4}{3}\Theta+\frac{1}{\tau}\right)P+\frac{8}{5}\,p\,\sigma\left[1+\nu_0\,\left(\frac{P}{p}\right)^2\right] \;=\; 0, \tag{52}$$

where in (52) $\nu_0 = 5/4,\ 25/32$, respectively, for the ideal gas and Stefan-Boltzmann laws (we examine the truncated equation (21) further ahead). The transport equation (52) is an evolution equation for P imposed by thermodynamic theories external (even if coupled) to General Relativity. On the other hand, there are evolution equations for P that follow from the field equations, for example, by evaluating $\dot{P}$ from (8) and eliminating $\dot{Y}/Y$ and $\dot{Y}'/Y'$ with the help of (10) and (11), leading to

$$\dot{P} \;+\; \left(5\sigma+\frac{4}{3}\Theta\right)P \;-\; 2\,p\,\sigma \;=\; 0, \tag{53}$$

an exact equation that must be satisfied by the generalized LTB models. Obviously, an equation like (53) does not exactly coincide with (52), and so compatibility between these models and Extended Irreversible Thermodynamics requires finding an appropriate expression for τ that should make (53) consistent with the transport equation. Comparing (52) (with $\nu_0 = 5/4$, ideal gas) with the evolution equation (53), we can see that both equations coincide if we identify

$$\tau = \frac{-p\,P}{\sigma\ \left\{2\,P^2 + \frac{18}{5}\,p^2 - 5\,p\,P\right\}} = -\frac{1}{4\sigma}\ \frac{\left[4+3\Delta_i^{(r)}-4\Gamma\right]\left[4+3\Delta_i^{(r)}-\Gamma\right]}{\frac{4}{5}\left[4+3\Delta_i^{(r)}+\frac{13}{32}\Gamma\right]^2+\frac{171}{256}\Gamma^2}, \tag{54}$$

This expression is justified as long as it behaves as a relaxation parameter for the interactive matter-radiation mixture in the theoretical framework of EIT as we will explain bellow.

It is also useful to compute the collision times for Thomson and Compton scattering (the dominant radiative processes in the radiative era)

$$t_{_\gamma} = \frac{1}{c\sigma_T n_e}, \quad t_c = \frac{m_e c^2}{k_{_B} T}\, t_{_\gamma}\,, \tag{55}$$

where σ_T is the Thomson scattering cross section, m_e is the electron mass and n_e is the number density of free electrons, a quantity obtained from Saha's equation, leading to

$$t_\gamma = \frac{1}{2c\,\sigma_{_T} n^{(m)}}\left[1+\left(1+\frac{4h^3 n^{(m)}\exp\left(B_0/k_{_B}T\right)}{\left(2\pi\,m_e k_B T\right)^{3/2}}\right)^{1/2}\right], \tag{56}$$

where B_0 and h are respectively the hydrogen atom binding energy and Planck's constant. For details of the derivation of (54) see [16].

The restrictions for physical acceptability and thermodynamic consistency were discussed in [26] and [16]. We provide a summary in the following list:

- (i) Positive definiteness and monotonicity of $\rho^{(r)}$ and $\rho^{(m)}$, and $|P| \ll p$.
- (ii) Regularity condition
$$\Gamma\ >\ 0, \tag{57}$$
which prevents the occurrence of a shell crossing singularity [17].
- (iii) Positive definiteness of τ and $\dot{s}$ (consistent with positive entropy production). The sign of $\dot{s}$ (see (51)) depends only on the sign of τ.
- (iv) Concavity and stability of s by requiring
$$\dot{\tau} > 0, \quad \frac{\ddot{s}}{\dot{s}} = \frac{2\sigma\Gamma}{3\Psi\Phi}\,\frac{\left\langle\rho_i^{(r)}\right\rangle}{\rho_i^{(r)}}\left[1+\frac{\left\langle\rho_i^{(r)}\right\rangle}{3\rho_i^{(r)}}\right]-\frac{\dot{\tau}}{\tau} < 0. \tag{58}$$
- (v) Appropriate behavior of the relaxation time τ.

  During the radiative era the Thompson time t_γ must be smaller than the expansion (or Hubble) time, approximately defined by $t_H = 3/\Theta$. The decoupling hypersurface is then defined by $t_\gamma = t_H$. From (iii) and (iv) above [15], [16], τ must be a positive and monotonously

increasing function (if the fluid expands), then it must also be (during the interactive period) qualitatively similar but larger than the microscopic timescales of the various photon-electron interactions occurring in the radiative era. It is usually assumed that τ is of the order of magnitude of a collision time, and as matter and radiation decouple all these timescales must overtake the Hubble expansion time $t_{_H} = 3/\Theta$. A physically reasonable τ should be comparable in magnitude to t_γ near decoupling and must have an analogous qualitative behavior to t_γ. Therefore τ must be smaller than t_H during the radiative era and then must overtake it at the decoupling hypersurface.

In the following list we test the previous restrictions (i) to (v) under the assumption of the small contrasts approximation (39), (40) and (41), together with either adiabatic (49) or quasi-adiabatic (46) initial conditions:

- (i) For a wide range of initial conditions all these quantities have a physically meaningful behavior as indicated in Figure. 1.
- (ii) Figure. 2 shows that Γ is positive and almost equal to unity for most of the evolution range of y. For the small range defined by $10^{-5} \leq |\Delta_i^{(S)}| \leq 10^{-3}$ and $\log_{10}(y) \leq 1$ the plot of Γ is distinct for lumps and voids.
- (iii) and (iv). Figure. 3 shows that $\tau > 0$, $\dot{s} > 0$ are satisfied for quasi-adiabatic initial conditions. From Figure. 3 it can also be seen that τ is a monotonically increasing function and so $\ddot{s} < 0$. The same results hold for adiabatic conditions (see Figure. 6a).
- (v) Under adiabatic initial conditions ($|\Delta_i^{(s)}| = 0$), τ does not have the required behavior described above (never overtakes $t_{_H}$). This is shown in Figure. 6a. We find that the desired behavior of τ is encountered for quasi-adiabatic initial conditions. As shown in Figure. 4, this happens in the elliptic case for the values $10^{-5.5} \leq |\Delta^{(s)}| \leq 10^{(-3)}$, compared to $|\Delta^{(S)}| \leq 10^{(-8)}$ in the hyperbolic (not shown) and parabolic case [26]. Finally, we plot the ratios $\tau/t_{_H}$, $t_c/t_{_H}$ and $t_\gamma/t_{_H}$ in Figure. 5 showing that these times have a physically reasonable and consistent behavior.

Furthermore, we examine the truncated transport equation for the generalized LTB models that follows by inserting (50) into (21)

$$\dot{P} + \frac{1}{\tau}\,P + \frac{8}{5}\,p\,\sigma = 0, \tag{59}$$

A comparison between (52) and (59) shows that the full equation is equivalent to the truncated one if $P/p \ll 1$ and $1/\tau \ll 4\Theta/3$. The first condition simply requires small deviations from equilibrium (as shown by (51)) and is compatible with energy conditions [15]. However, the condition $4\tau \gg 3/\Theta$ is more problematic and is only reasonable after matter and radiation have decoupled, hence, under the assumptions underlying the thermodynamic study of the models, the truncated equation cannot describe the interacting period nor the decoupling process. This shortcoming of (21) has been discussed previously for the parabolic case [26]. As shown by Figure. 6b, this situation holds also for the elliptic and hyperbolic cases.

In order to complete the analysis we compute the Jeans mass associated to the initial conditions of the case $\Delta_i^{(s)} \neq 0$. This mass is given by[9],[8],[6]

$$M_{J} = \frac{4\pi}{3} m\, n^{(m)} \left[\frac{c^4 \pi C_s^2}{G\,(\rho + p)}\right]^{3/2} = \frac{4\pi}{3} \frac{c^4 \chi_i \Gamma^{1/2}}{\sqrt{\rho_i^{(r)}}} \left[\frac{\pi y^2 \Psi}{3G\left(\Psi + \frac{3}{4}\chi_i y\right)^2}\right]^{3/2}, \tag{60}$$

where ρ, p, $n^{(m)}$ are given by (14), (17) and (40), $\chi_i = \rho_i^m/\rho_i^r$, Ψ and Γ follow from (33) and C_s is the speed of sound, which for the equation of state (14), has the form

$$C_s^2 = \frac{c^2}{3}\left[1 + \frac{3\rho^{(m)}}{4\rho^{(r)}}\right]^{-1}, \quad \rho^{(m)} = mc^2 n^{(m)}, \quad \rho^{(r)} = 3n^{(r)} k_{B} T. \tag{61}$$

Evaluating (60) for $y = y_D \approx 10^{2.4}$, $\epsilon \approx 1/\chi_i \approx 10^3$ and $\rho_i^{(r)} \approx a_{B} T_i^4 \approx 7.5 \times 10^9\,\text{ergs/cm}^3$, yields $M_{J} \approx 10^{49}$gm, or approximately $10^{16} M\odot$. This value coincides with the Jeans mass obtained for perturbative models dominated by baryons in the radiative era as decoupling is approached. Finally, it is worthwhile remarking that we are dealing with an epoch where a cosmological constant or a quintessence component should be clearly subdominant [27] and ignored.

IX. CONCLUSION

We have derived and discussed important generic properties of a class of exact solutions of Einstein's equations that generalize the famous LTB solutions with a dust source to an imperfect fluid with anisotropic pressure Π_{ab}. The integration of the field equations does not involve Π_{ab}, though once this integration is done this pressure becomes also determined (up to initial conditions). The issue regarding the compatibility of Π_{ab} and its evolution law with the physical assumptions underlining the models have been addressed. These models provide a physically plausible hydrodynamical description of cosmological matter-radiation mixture in the radiative era, between nucleosynthesis and decoupling. By assuming small initial density contrasts (consistent with small deviations from equilibrium), we have shown that the state variables of the models are qualitatively and formally analogous to that of adiabatic perturbations on a FLRW background. We have also found two classes of initial conditions, based on well defined initial contrast functions, that are formally equivalent to the definition of adiabatic and quasi-adiabatic perturbations on initial value functions $\rho_i^{(m)}$, $\rho_i^{(r)}$, ${}^{(3)}\mathcal{R}_i$. In particular, we showed that for quasi-adiabatic initial conditions the models are thermodynamically consistent and physically acceptable. However, this consistency does not hold for the truncated transport equation (as shown also in [26]), only for the full transport equation of Extended Irreversible Thermodynamics the relaxation time of shear viscosity has the appropriate physical behavior.

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[28] This is a package which runs within Maple. It is entirely distinct from packages distributed with Maple and must be obtained independently. The GRTensorII software and documentation is distributed freely on the World-Wide-Web from the address http://grtensor.org

APPENDIX A: INTEGRATION OF THE FRIEDMANN EQUATION (22).

The quadrature (22) depends on the sign of $\kappa = K/Y_i^2$. The integral yields either an implicit function $ct = Z(y, \mu, \omega, \kappa)$ where Z is defined by (32) (the "canonical solution"), or a parametric solution having the form $y = y(\eta, r)$, $t = t(\eta, r)$. The canonical solution of the equation equivalent to (22) found in the literature for the LTB dust case [1],[2], is usually given in the form $c[t - t_0(r)] = Z(y, \mu, \kappa)$, where the function $t_0(r)$ comes as an integration constant. Instead, we will provide the canonical solutions in the form $c(t - t_i) = Z - Z_i$, so that we can identify $t_0(r) = t_i + Z_i$. The function $t_0(r)$ can also be identified as the "big bang time". Only for $\kappa = 0$ the canonical solution can be inverted as $y = y(t, r)$ by solving a cubic

equation, however the resulting expression is too cumbersome and will not be given.

All variables introduced in previous sections will be made fully determined as functions of y and of r, with the dependence on r given through the initial value functions and their initial contrast functions. As usual in the study of dust LTB solutions, we will classify the integrals (32) in three cases: "parabolic" ($\kappa = 0$), "elliptic" ($\kappa > 0$) and "hyperbolic" ($\kappa < 0$). We shall examine each case separately below.

1. Parabolic case.

If $\kappa = 0$, so that ${}^{(3)}\mathcal{R}_i = \Delta_i^{(k)} = 0$, then (32) leads to the canonical form

$$c(t-t_i) = Z - Z_i = \frac{4}{3\sqrt{\mu}}\left[\sqrt{2y+\epsilon_1}\,(y-\epsilon_1) - \sqrt{2+\epsilon_1}\,(1-\epsilon_1)\right] \qquad \text{(A1)}$$

where ϵ_1 was defined in (25). The solution in terms of the parameter η is

$$y \;=\; \frac{1}{2}\left[\frac{\eta}{\mu} - \epsilon_1\right], \quad c(t-t_i) \;=\; \frac{\sqrt{\eta}}{6\mu}\left[\frac{\eta}{\mu} - \epsilon_1\right] \;-\; Z_i \qquad \text{(A2)}$$

The functions A, B in (36) have the form

$$A \;=\; \frac{1}{3y^2}\left[\frac{\sqrt{2y+\epsilon_1}}{\sqrt{2+\epsilon_1}}(1-4\epsilon_1-8\epsilon_1^2) - (y^2 - 4\epsilon_1 y - 8\epsilon_1^2)\right],$$

$$B \;=\; \frac{2\epsilon_1}{y^2}\left[(1+\epsilon_1)\frac{\sqrt{2y+\epsilon_1}}{\sqrt{2+\epsilon_1}} - (y+\epsilon_1)\right] \qquad \text{(A3)}$$

Notice that the limit $\rho_i^{(r)} \to 0$ (or $\epsilon_1 \to 0$) leads to the known parabolic solutions of the dust LTB solutions, though, the limit $\rho_i^{(m)} \to 0$ is singular.

2. Elliptic case

For $\kappa > 0$ the canonical integral (32) is

$$c(t-t_i) = Z - Z_i = \left| -\frac{\sqrt{Q}}{\kappa} + \frac{\mu}{\kappa^{3/2}}\arccos\left(\frac{Q_{,y}}{\sqrt{\lambda}}\right)\right|_{y=1}^{y} \qquad (A5)$$

$$Q = -\kappa y^2 + 2\mu y + \omega, \quad Q_{,y} = -2\kappa y + 2\mu, \quad \lambda \equiv 4(\mu^2 + \kappa\omega)$$

while the parametric solution is given by

$$y = \frac{\mu}{\kappa}\left[1 - \frac{\sqrt{\lambda}}{2\mu}\cos\eta\right] \qquad (A6a)$$

$$c(t-t_i) = \frac{\mu}{\kappa^{3/2}}\left[\eta - \frac{\sqrt{\lambda}}{2\mu}\sin\eta\right] - Z_i \qquad (A6b)$$

The functions A, B, C in (36) become

$$A = \frac{\mu}{\kappa y^2}\left[\frac{(\eta_i - \eta)\sqrt{Q}}{\sqrt{\kappa}} + \frac{4\mu^2 + \lambda + 4\mu\omega}{\lambda}\,\frac{\sqrt{Q}}{\sqrt{Q_i}} - \frac{(4\mu^2+\lambda)y + 4\mu\omega}{\lambda}\right] \qquad (A7a)$$

$$B = \frac{2\omega}{\lambda y^2}\left[(\omega+\mu)\frac{\sqrt{Q}}{\sqrt{Q_i}} - (\omega+\mu y)\right] \tag{A7b}$$

$$C = -\frac{3\mu\sqrt{Q}(U_i - U)}{2\kappa^{3/2}y^2} + \frac{\left[(\lambda - 4\mu\omega)\kappa - (2\mu+\omega)(4\mu^2+2\lambda)\right]\sqrt{Q}}{2\kappa\lambda y^2\sqrt{Q_i}} -$$

$$\frac{\left[(\lambda - 4\mu\omega)\kappa y - (2\mu y+\omega)(4\mu^2+2\lambda)\right]\sqrt{Q_i}}{2\kappa\lambda y^2\sqrt{Q_i}} \tag{A7c}$$

$$\eta = \arccos\left(-\frac{Q_{,y}}{\sqrt{\lambda}}\right)$$

As with the parabolic case, we obtain the dust LTB elliptic solution in the limit $\omega \to 0$. It is also interesting to compare the evolution of y as function of η with the dust case. This evolution is also time-symmetric, but it begins at the value $\eta_{BB} = \arccos(2\mu/\sqrt{\lambda}) > 0$, instead of $\eta_{BB} = 0$. The maximal value of y as the fluid bounces is $y_{max} = (\mu/\kappa)(1+\sqrt{\lambda}/2\mu)$, larger than the value for the dust case $y_{max} = (\mu/\kappa)$.

3. Hyperbolic case

For $\kappa < 0$ the canonical integral (32) is

$$c(t-t_i) = Z - Z_i = \left|\frac{\sqrt{Q}}{|\kappa|} - \frac{\mu}{|\kappa|^{3/2}}\operatorname{arccosh}\left(\frac{Q_{,y}}{\sqrt{\lambda}}\right)\right|_{y=1}^{y}, \quad \lambda > 0 \tag{A8a}$$

$$c(t-t_i) = Z - Z_i = \left|\frac{\sqrt{Q}}{|\kappa|} - \frac{\mu}{|\kappa|^{3/2}}\ln\left(\frac{Q_{,y}}{2\sqrt{|\kappa|}} + \sqrt{Q}\right)\right|_{y=1}^{y}, \quad \lambda \le 0 \tag{A8b}$$

$$Q = |\kappa|y^2 + 2\mu y + \omega, \quad Q_{,y} = 2|\kappa|y + 2\mu, \quad \lambda \equiv 4\mu^2 - 4|\kappa|\omega$$

while the parametric solution is given by

$$y = \frac{\mu}{|\kappa|}\left[\frac{\sqrt{\lambda}}{2\mu}\cosh\eta - 1\right] \tag{A9a}$$

$$c(t-t_i) = \frac{\mu}{|\kappa|^{3/2}}\left[\frac{\sqrt{\lambda}}{2\mu}\sinh\eta - \eta\right] - Z_i \tag{A9b}$$

The functions A, B, C in (14) become

$$A = \frac{\mu}{\kappa y^2}\left[\frac{(\eta_i - \eta)\sqrt{Q}}{\sqrt{|\kappa|}} + \frac{4\mu^2+\lambda+4\mu\omega}{\lambda}\frac{\sqrt{Q}}{\sqrt{Q_i}} - \frac{(4\mu^2+\lambda)y + 4\mu\omega}{\lambda}\right] \tag{A10a}$$

$$B = \frac{2\omega}{\lambda y^2}\left[(\omega+\mu)\frac{\sqrt{Q}}{\sqrt{Q_i}} - (\omega+\mu y)\right] \tag{A10b}$$

$$C = -\frac{3\mu\sqrt{Q}(\eta_i - \eta)}{2|\kappa|^{3/2}y^2} + \frac{\left[(\lambda - 4\mu\omega)\kappa - (2\mu+\omega)(4\mu^2+2\lambda)\right]\sqrt{Q}}{2\kappa\lambda y^2\sqrt{Q_i}} -$$

$$\frac{\left[(\lambda-4\mu\omega)\kappa y-(2\mu y+\omega)(4\mu^2+2\lambda)\right]\sqrt{Q_i}}{2\kappa\lambda y^2\sqrt{Q_i}} \qquad (A10c)$$

$$\eta = \operatorname{arccosh}\left(-\frac{Q_{,y}}{\sqrt{\lambda}}\right)$$

As with the parabolic case, we obtain the dust LTB elliptic solution in the limit $\omega \to 0$.

APPENDIX B: PARTICULAR CASES.

Expressing initial value functions in terms of the initial averages and contrast functions defined in sections III and IV greatly facilitates the description of the correspondence to the following particular cases:

- **Dust limit.** If $\rho_i^{(r)} = \langle\rho_i^{(r)}\rangle = 0$, then $\omega = 0$ and $Q = (-\kappa y + 2\mu)y$, hence we have: $\rho = \rho^{(m)} = \rho_i^{(m)}/(y^3\Gamma)$ and $p = P = 0$, and so the solutions reduce to the "usual" LTB solutions with a dust source. The initial density and curvature contrasts $\Delta_i^{(m)}$, $\Delta_i^{(k)}$ have the same interpretation as in the general case. Notice that $B = 0$ (from (14a)) and that the auxiliary functions Ψ, Φ are irrelevant in this sub-case.

- **FLRW limit.** Homogeneity along the initial hypersurface follows by demanding $\Delta_i^{(m)} = \Delta_i^{(r)} = \Delta_i^{(k)} = 0$, leading to $\rho_i^{(m)} = \langle\rho_i^{(m)}\rangle$, $\rho_i^{(r)} = \langle\rho_i^{(r)}\rangle$, ${}^{(3)}\mathcal{R}_i = \langle{}^{(3)}\mathcal{R}_i\rangle$, so that $\rho_i^{(m)}$, $\rho_i^{(r)}$, ${}^{(3)}\mathcal{R}_i$ are constants. Equation (22) implies then that μ, ω, κ are also constants, so that M, $W \propto Y_i^3$ and $K \propto Y_i^2$. Therefore y obtained by integrating (22) must be a function of t only and $Y = R(t)Y_i$. From (33) to (35) we have: $\Gamma = \Psi = 1$ and $\Phi = 0$ leading to $\rho = \rho(t)$, $p = p(t)$, $P = 0$ and $\sigma = 0$, $\Theta = 3\dot{y}/y = 3\dot{R}/R$. Thus the particular case in which initial density and curvature contrasts vanish is the FLRW limit of the solutions, a FLRW cosmology where ρ, p satisfy the "dust plus radiation" relation (6).

- **Vaidya limit.** If $\rho_i^{(m)} > 0$, $\rho_i^{(r)} > 0$ for $0 \le r < r_b$ but vanish for $r \ge r_b$, then $\langle\rho_i^{(m)}\rangle$ and $\langle\rho_i^{(r)}\rangle$, as well as M and W are constants for $r \ge r_b$ and we have (along this range) a sub-case of the Vaidya metric characterized by the mass function $m = M(r_b) + W(r_b)Y_i/Y$. This particular Vaidya solution is the "exterior" field for the models derived in this paper, generalizing the Schwarzschild exterior of LTB dust solutions. Notice that as the fluid expands the Vaidya mass function, m, tends to a constant Schwarzschild mass given by $M(r_b)$. More details are provided in [15].

The solutions derived in this paper can be smoothly matched along a comoving hypersurface marked by $r = r_b$ with their FLRW and Vaydia sub-cases. The matching conditions are discussed in detail in [15]. Just as LTB dust solutions can be generalized to the Szekeres dust solutions without isometries, LTB solutions with an imperfect fluid source can be generalized to the Szafron-Szekeres metrics with an imperfect fluid source[15]. Even without spherical symmetry, these solutions can still be smoothly matched to the FLRW and Vaydia sub-cases along a spherical comoving boundary [15].

FIG. 1: (a) Logarithmic plot of the densities ratio. These densities are positive definite and decreasing functions of $\log_{10}(y)$; The plot also shows that for the early stage the radiation density dominates while for later stages the matter density dominates. (b) The ratio P/p is very small complying with energy condition. P is negative/positive for lumps/voids respectively. As we are dealing with small initial density contrasts defined by (39), (40) and (41), we used for all the plots $\epsilon_1 \approx 2\rho_i^{(r)}/\rho_i^{(m)} = 10^3$ and $\epsilon_2 \approx (c^4/8\pi G)({}^{(3)}\mathcal{R}_i/\rho_i^{(m)}) = 10^{-7}$.

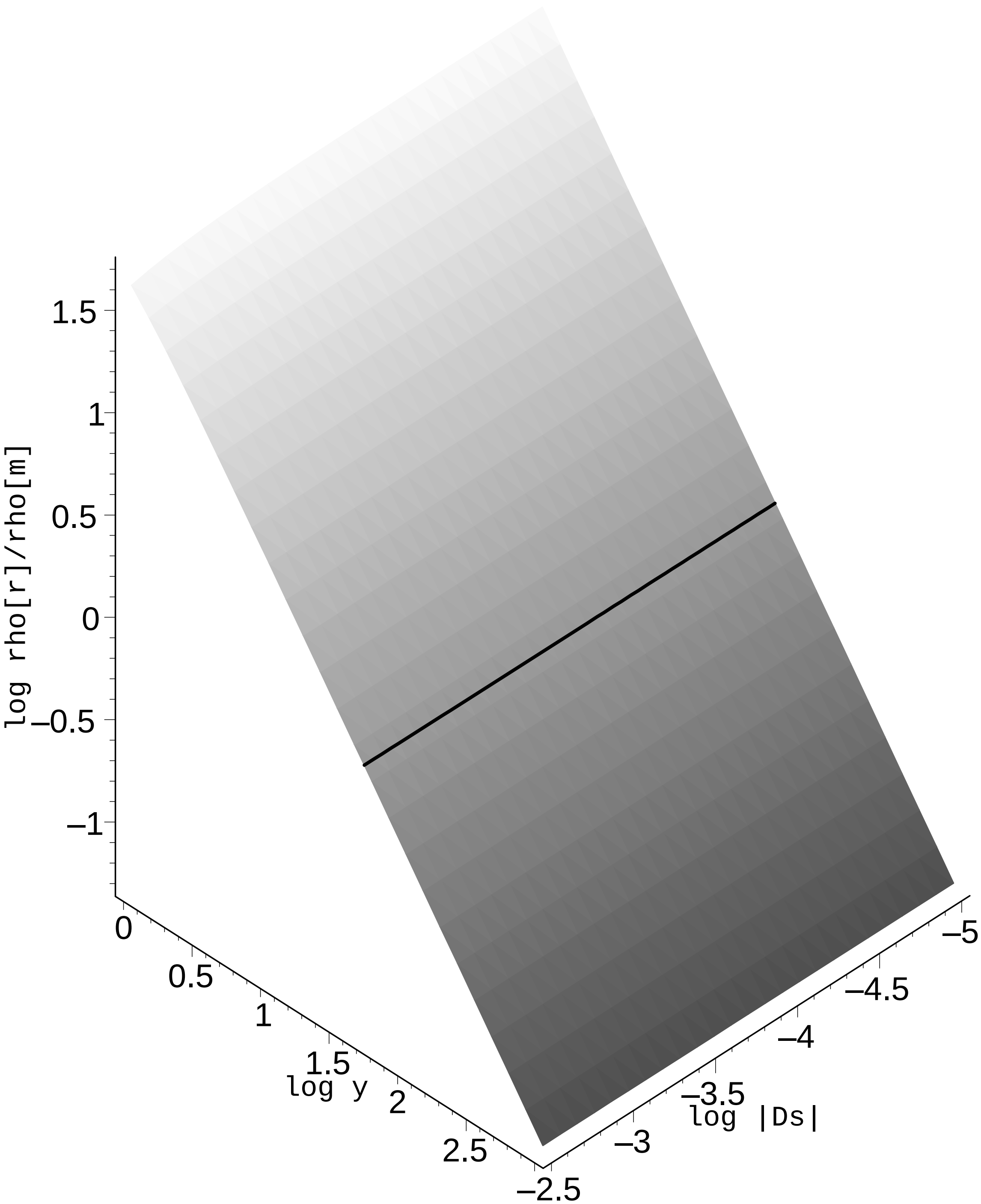
FIG. 2: The function Γ given by (33) in terms of $\log_{10}(y)$ and $\log_{10}(|\Delta_i^{(s)}|)$, from the initial hypersurface $y = 1$ corresponding to $T_i = 10^6$ Kelvin. Γ is positive as required, it is almost equals to one for all the evolution range. (a) The lumps case, (b) the voids case.

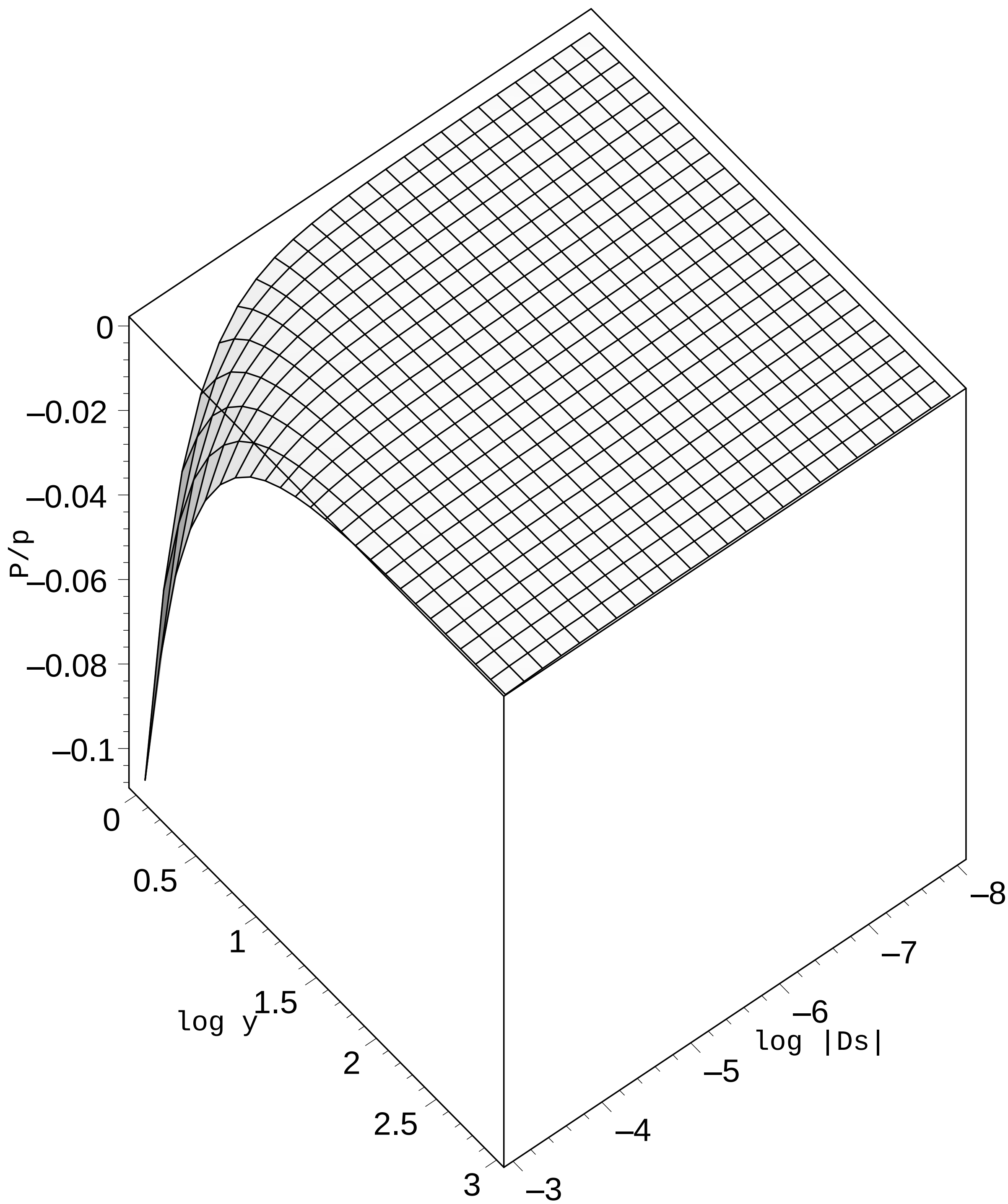
FIG. 4: Plot of $\log_{10}(\tau/t_H)$ in terms of $\log_{10}(y)$ and $\log_{10}(|\Delta_i^{(s)}|)$; (a) for $10^{-8} \leq |\Delta_i^{(s)} < 10^{-5.5})$ the relaxation time is not initially smaller than the Hubble time ; (b) For the range $10^{-5.5} \leq |\Delta_i^{(s)}| \leq 10^{(-3)}$ the relaxation time is initially smaller than the Hubble time, and it overtakes it at about $y = 10^{2.4}$ corresponding to $T_D = 4000K$ satisfying the required behavior.

FIG. 5: The figure depicts the ratios $\log_{10}(t_c/t_H)$, $\log_{10}(\tau/t_H)$ and $\log_{10}(t_\gamma/t_H)$ versus $\log_{10}(y)$ for $\Delta_i^{(r)} = 10^{-4}$ and $\Delta_i^{(s)} = 10^{-3}$. Initially, the ratio τ/t_H is far larger than t_γ/t_H but become comparable to it near the decoupling hypersurface defined by $t_\gamma = t_H$. Note the similarity in the qualitative behavior of these two ratios. This plot shows also how the Compton time is important for high temperatures (near $\log_{10}(y) = 0$) but rapidly overtakes t_H and so Compton scattering is no longer an efficient radiative process. Thompson scattering is very small initially but becomes the dominant process near decoupling. The relaxation time τ is a mesoscopic quantity that acts roughly as an "average" timescale for all these processes.

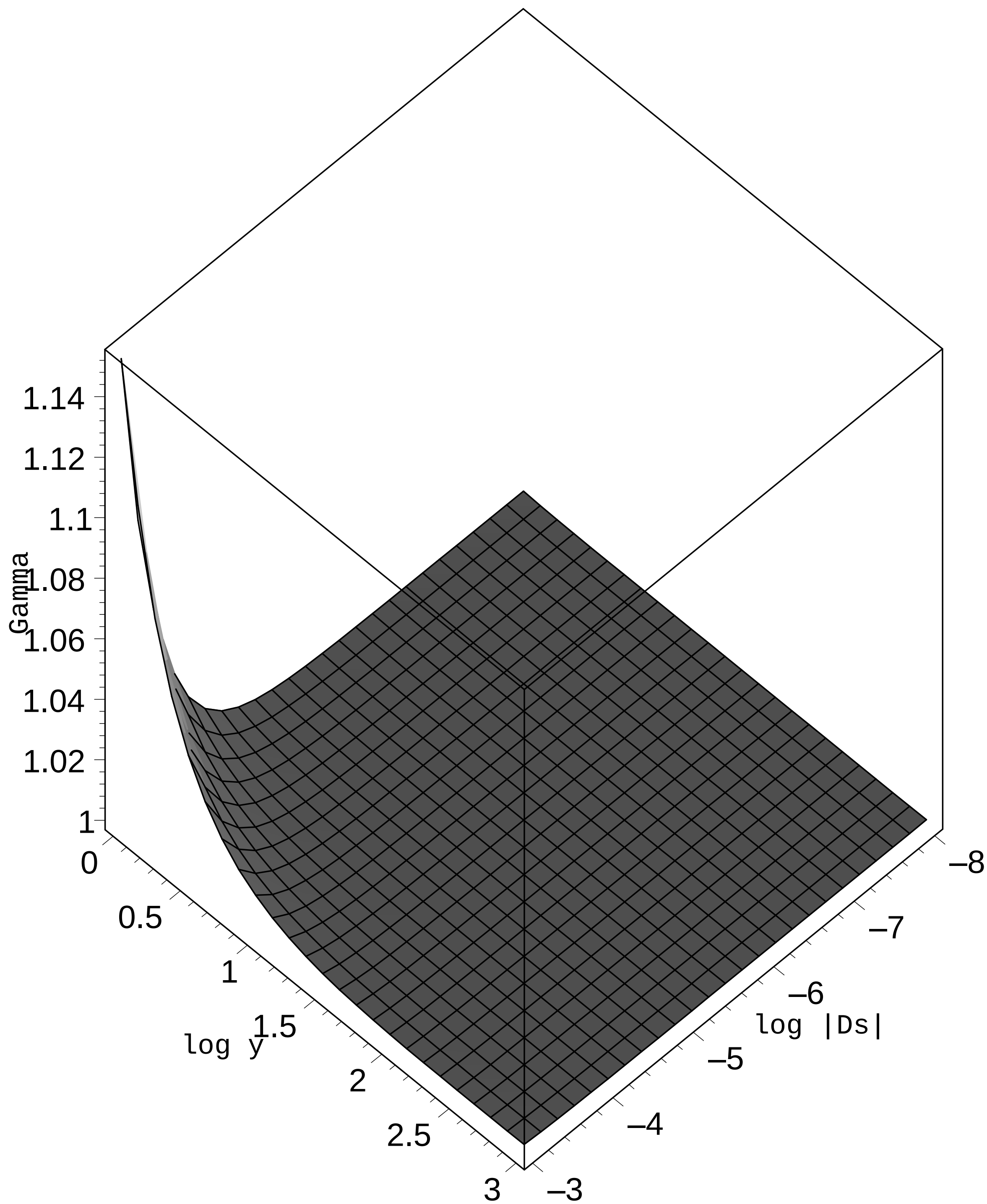
FIG. 6: Plot of $\log_{10}(\tau/t_H)$ in terms of $\log_{10}(y)$, (a) for adiabatic conditions $|\Delta_i^{(s)}| = 0$, τ/t_H is always smaller then 1, so τ does not overtake t_H, (b) For τ defined from the truncated equation (49c) τ/t_H is always smaller then 1 as well.

FIG. 3: Plot of $\log_{10}(\tau)$ in terms of $\log_{10}(y)$ and $\log_{10}(|\Delta_i^{(s)}|)$, τ is a positive definite and increasing function for $10^{-8} \leq |\Delta_i^{(s)}| \leq 10^{-3}$ and for all evolution range.

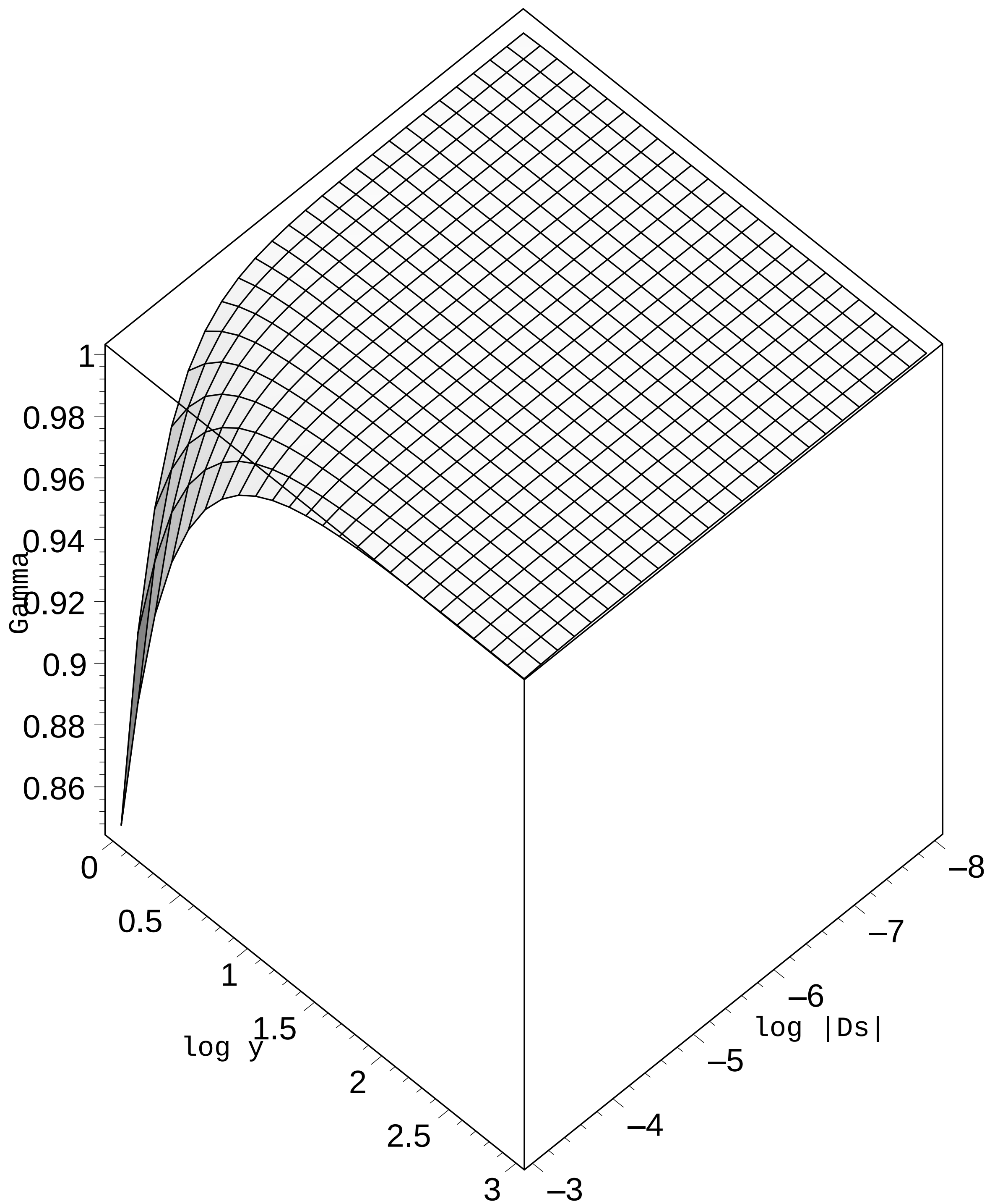

log rho[r]/rho[m]
1.5
1
0.5
0
–0.5
–1
0
0.5
1
1.5
2
2.5
log y
–2.5
–3
–3.5
–4
–4.5
–5
log |Ds|



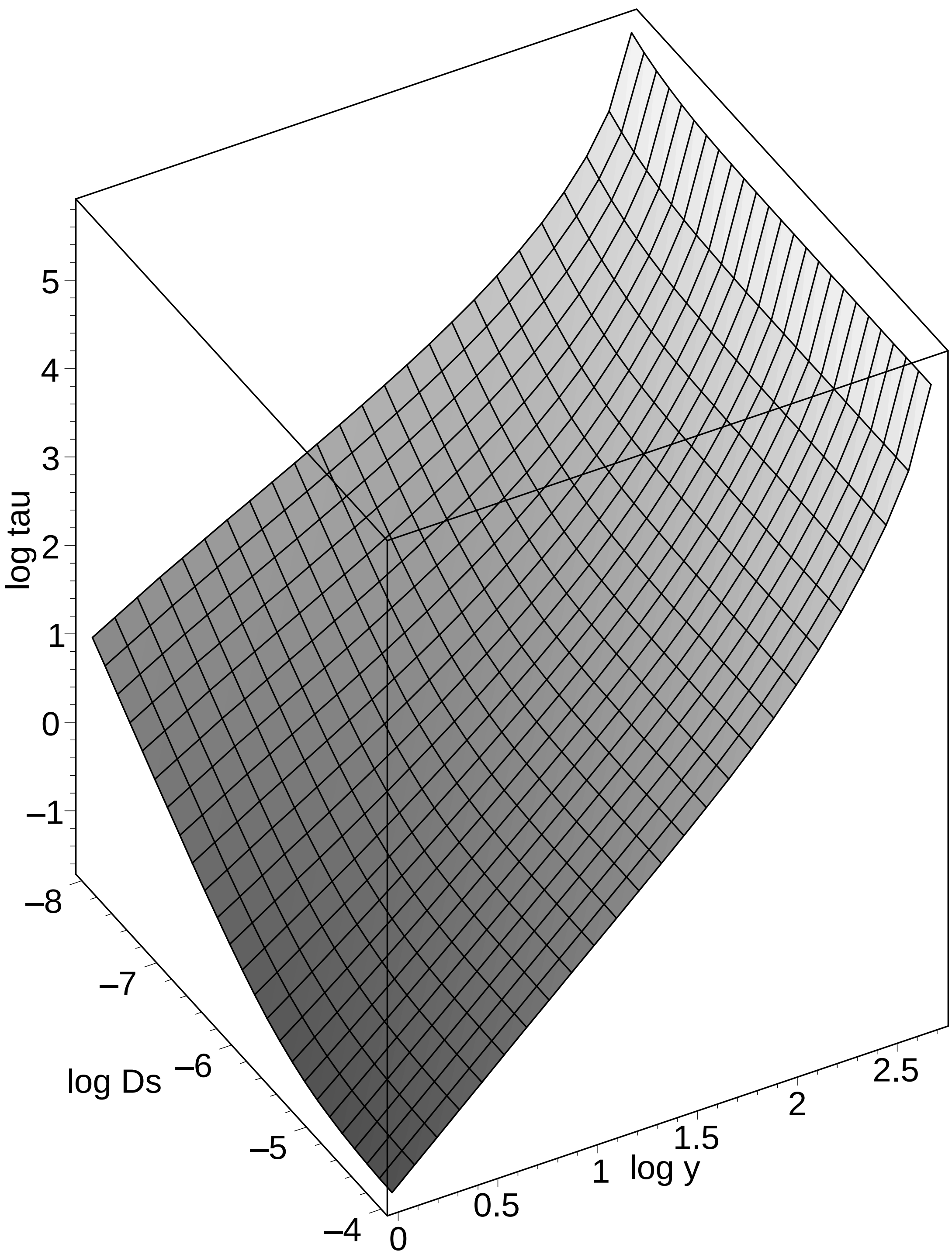
P/p
0
–0.02
–0.04
–0.06
–0.08
–0.1
log y
0
0.5
1
1.5
2
2.5
3
log |Ds|
–3
–4
–5
–6
–7
–8



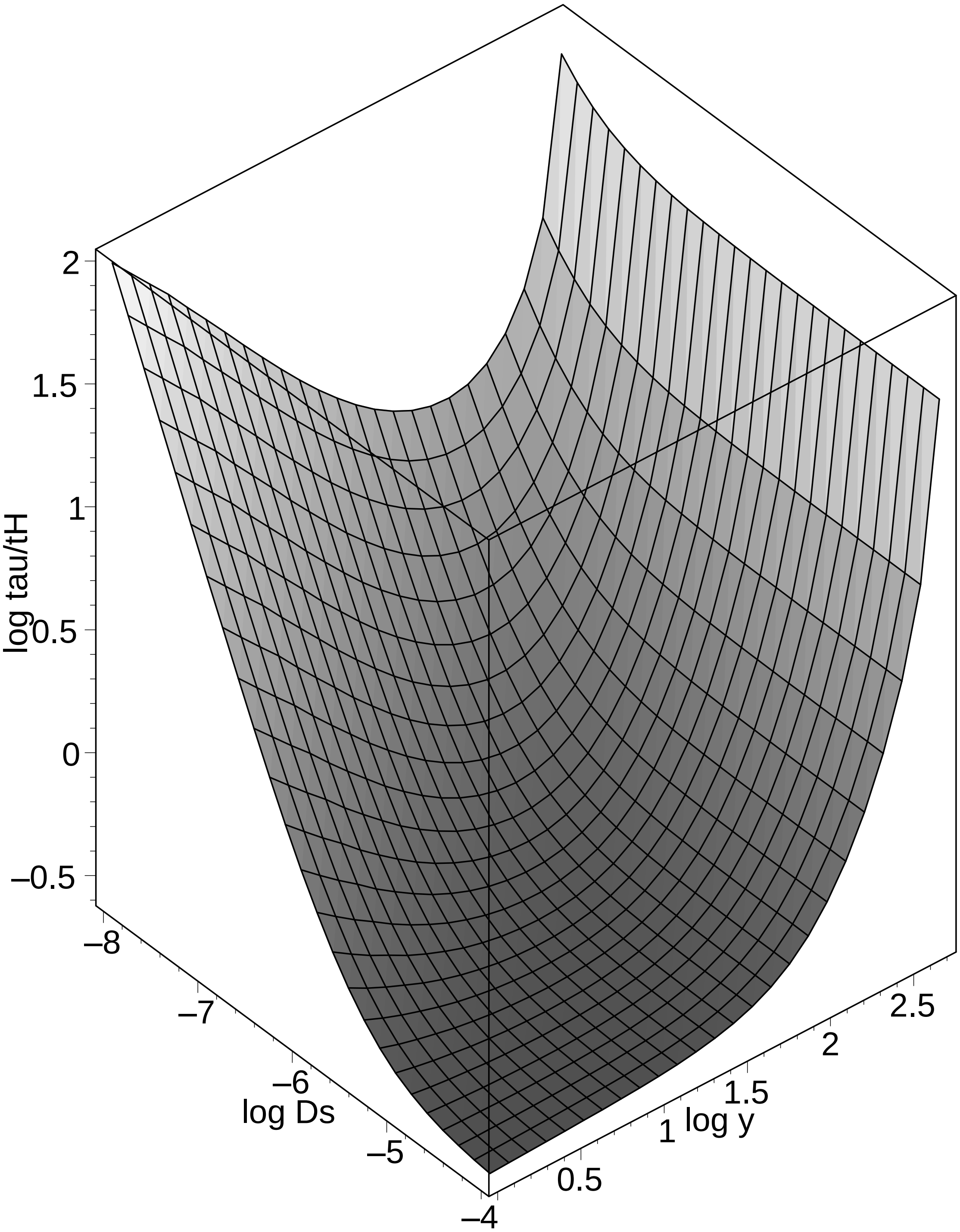
Gamma
1.14
1.12
1.1
1.08
1.06
1.04
1.02
1
0
0.5
1
1.5
2
2.5
3
log y
–3
–4
–5
–6
–7
–8
log |Ds|



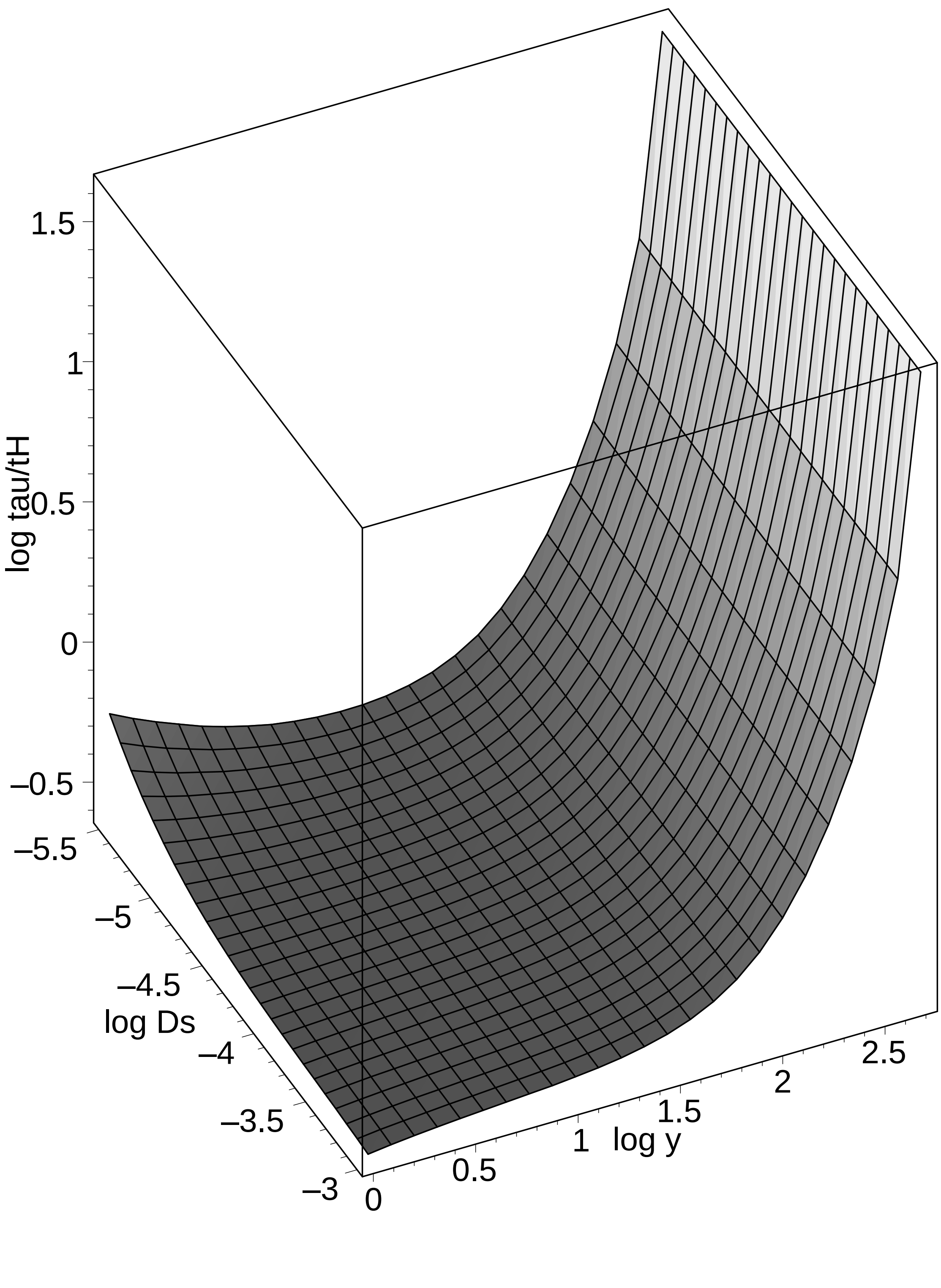
Gamma
1
0.98
0.96
0.94
0.92
0.9
0.88
0.86
0
0.5
1
1.5
2
2.5
3
log y
–8
–7
–6
–5
–4
–3
log |Ds|



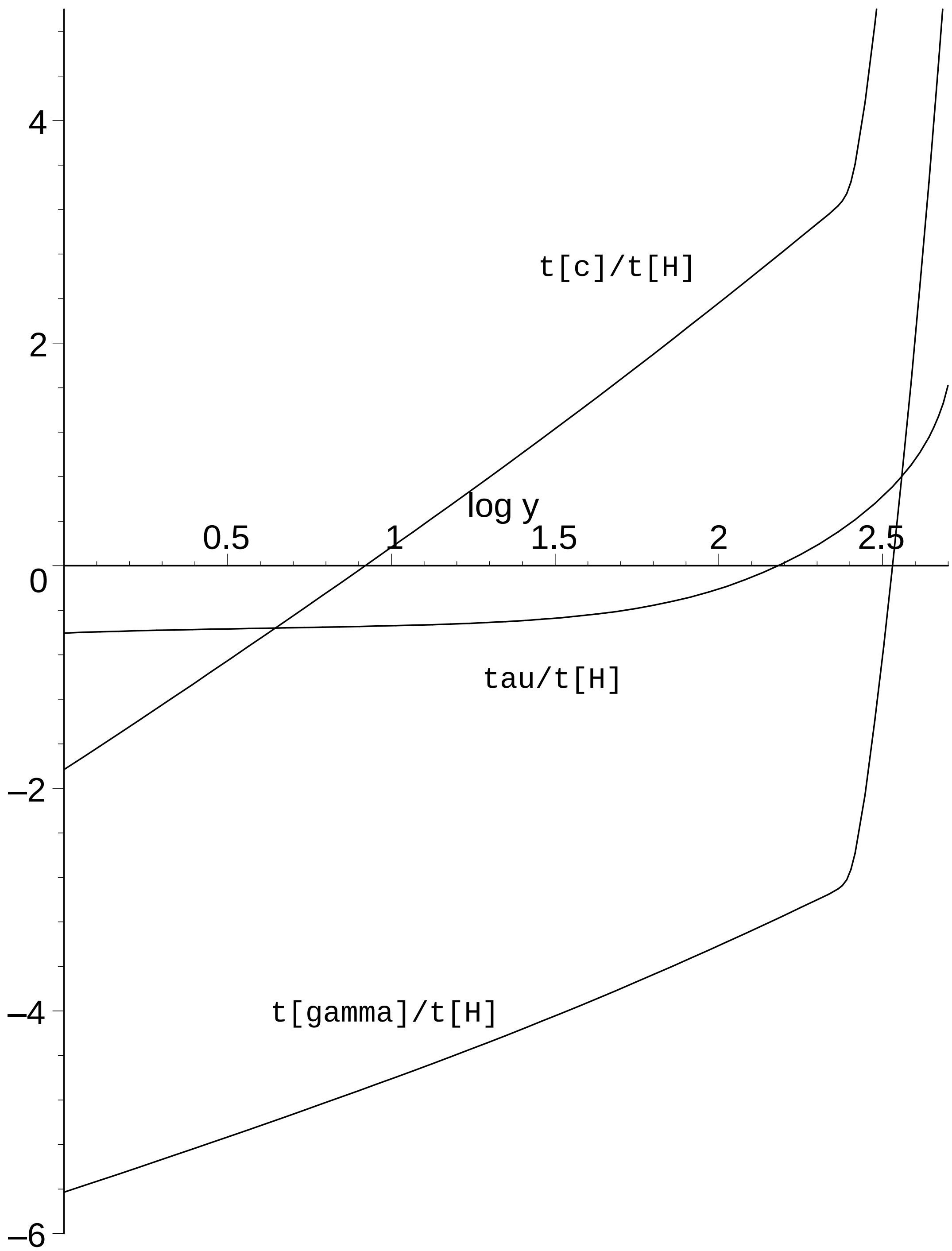
log tau
5
4
3
2
1
0
–1
–8
–7
–6
–5
–4
log Ds
0
0.5
1
1.5
2
2.5
log y



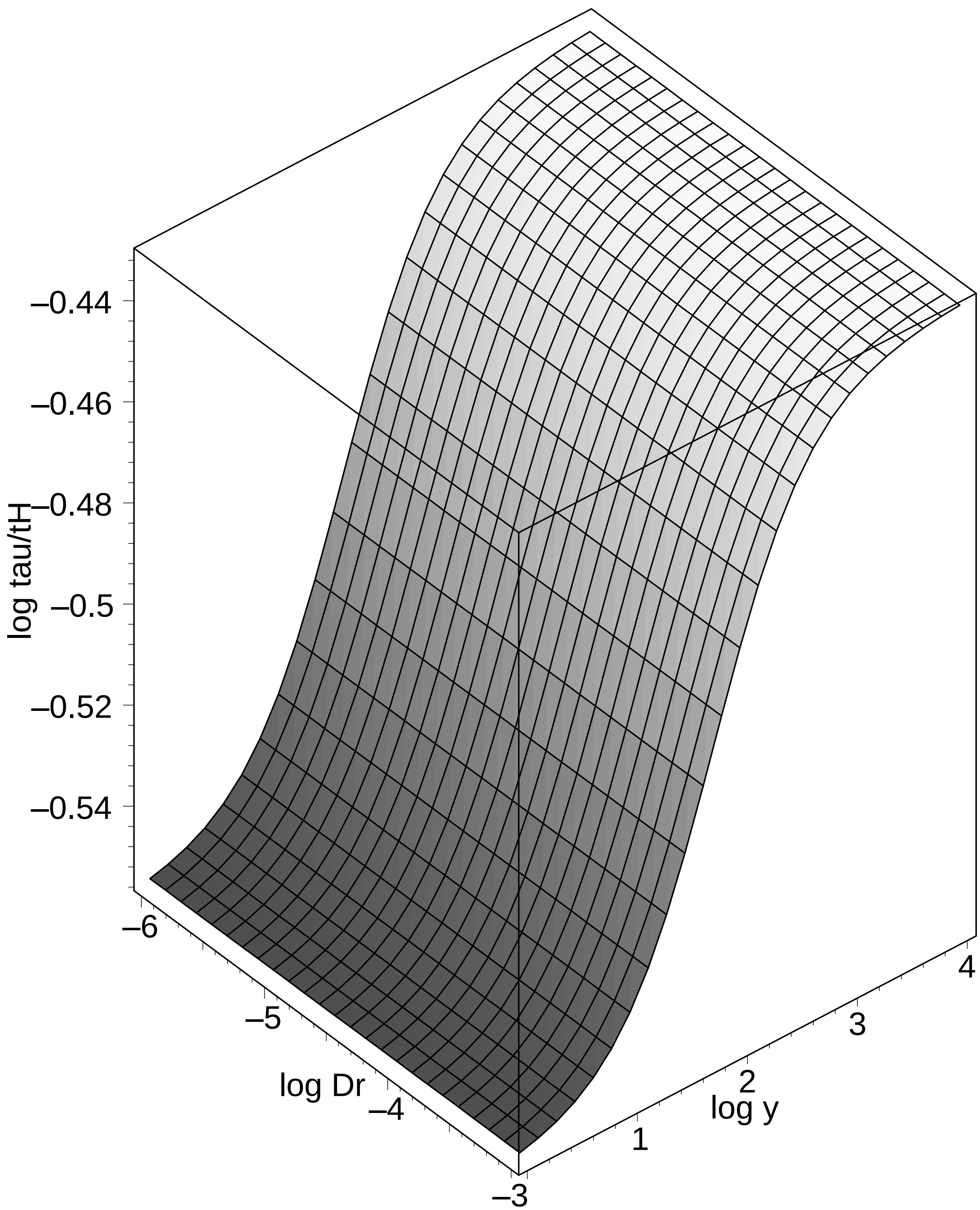
log tau/tH
2
1.5
1
0.5
0
–0.5
–8
–7
–6
log Ds
–5
–4
0.5
1
log y
1.5
2
2.5

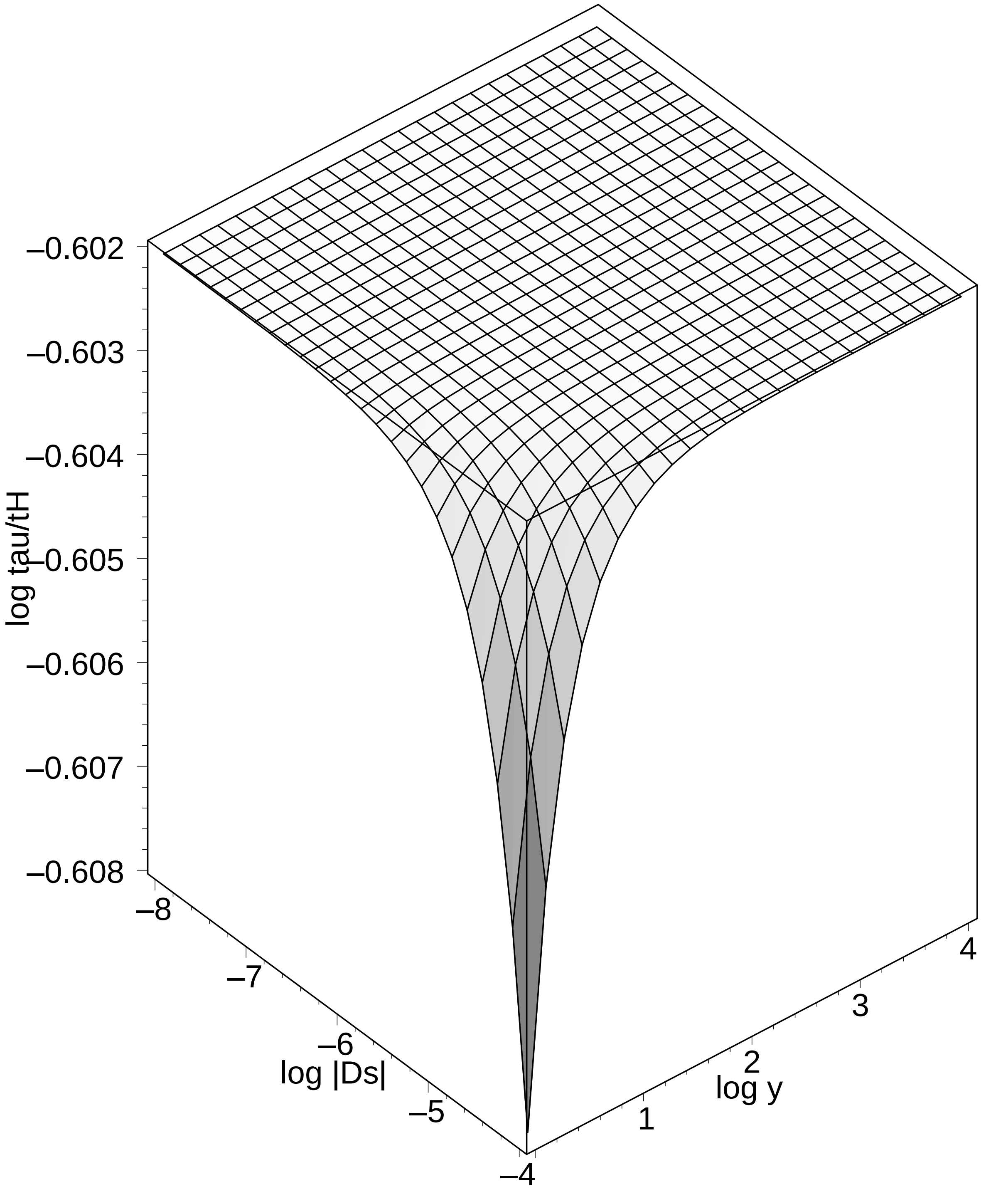


log tau/tH
1.5
1
0.5
0
–0.5
–5.5
–5
–4.5
log Ds
–4
–3.5
–3
0
0.5
1
log y
1.5
2
2.5



t[c]/t[H]
log y
0.5
1
1.5
2
2.5
4
2
0
–2
–4
–6
tau/t[H]
t[gamma]/t[H]


log tau/tH
–0.44
–0.46
–0.48
–0.5
–0.52
–0.54
–6
–5
log Dr
–4
–3
1
2
log y
3
4



log tau/tH
–0.602
–0.603
–0.604
–0.605
–0.606
–0.607
–0.608
log |Ds|
–8
–7
–6
–5
–4
log y
1
2
3
4